MATH 116 Calculus I for Engineering

Lecture notes

Sepehr Hajebi*

Fall 2025 – University of Waterloo

*This course offering involved multiple instructors, with Zack Cramer serving as the head instructor and course coordinator. Accordingly, while the exposition, examples, and presentation here are very much shaped by my own approach and style, these lecture notes follow the official course syllabus and draw on the structure of Zack's notes (available at `https://www.math.uwaterloo.ca/~zcramer/#math116`).

Contents

1 Functions

1. Sets and numbers.

- We write $\mathbb{R}$ for the set of all real numbers. Unless specified otherwise, the word "number" will always mean a real number.
- A set is specified by listing its elements or giving an expression inside curly brackets $\{\}$.
- The symbol $\in$ is read as "belongs to." For example, $x \in \mathbb{R}$ means "x is a real number."
- When describing sets, the symbol $|$ is read as "such that." For instance, $\{x \in \mathbb{R} \mid x \geq \pi\}$ denotes the set of all real numbers x such that x is greater than or equal to π.
- The notation $\blacksquare$ means "end of solution."

2. Intervals. For $a, b \in \mathbb{R}$ with $a \leq b$, we define:

- $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}, \quad [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$,
- $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}, \quad (a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$,
- $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}, \quad [a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$,
- $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}, \quad (-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$.

3. Functions. A function (for us) is a rule f assigning each input $x \in \mathbb{R}$ **exactly one** output $y \in \mathbb{R}$. We denote this by $f : \mathbb{R} \to \mathbb{R}$.

4. Ways to represent functions.

1. Equation: We write $y = f(x)$ to mean x is the input, also called the *independent variable*, and y is the output, also called the *dependent variable*. We also say that "y is a function of x."

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be the function with equation $y = f(x) = 1 - \sqrt{x}$.
Then $f(1) = 0$, $f(121) = -10$.

Two well-known types of functions that we can tell by their equations:

Polynomials: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, \ldots, a_0 \in \mathbb{R}$ are the *coefficients* with $a_n \neq 0$, and $n \geq 0$ is an integer called the *degree* of f.
Example. $f(x) = x^{17} - x^4 + 2x^3 - 3x^2 + 4$ is a polynomial of degree 17, but $f(x) = 1 - \sqrt{1 - \sqrt{x}}$ is **not** a polynomial.

Rational functions. $f(x) = \dfrac{p(x)}{q(x)}$, where p, q are polynomials (and $q \neq 0$).

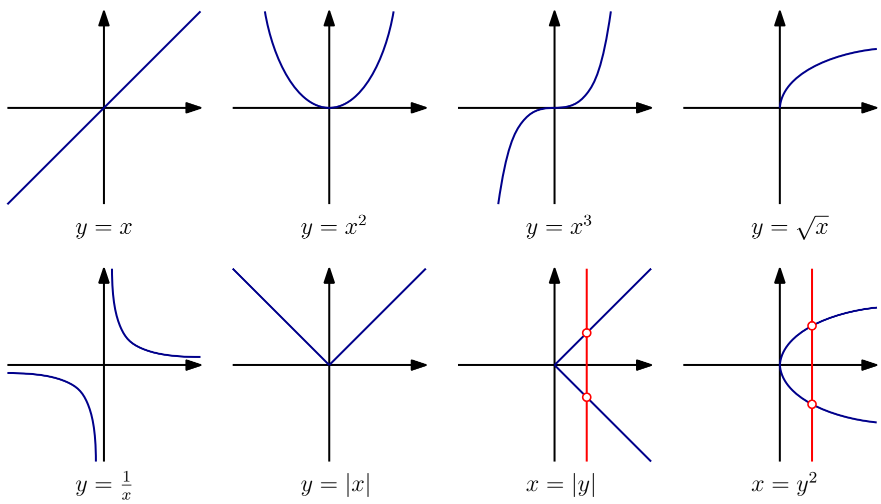
Example. $f(x) = \dfrac{x - x^4}{2 - x^{11}}$ is a rational function, and so is $f(x) = \dfrac{x}{1 - x} - \dfrac{1 - x}{x}$ after simplification.

2. Graph: The set of all points (x, y) in the plane that satisfy $y = f(x)$.

Here are some familiar equations with their graphs (see the above figure). Some indicate y as a function of x, and some do **not**.

- $y = x$ (the *identity* function),

- $y = x^2$ (the *square* function),
- $y = x^3$ (the *cube* function),
- $y = \sqrt{x}$ (the *square root* function),
- $y = \dfrac{1}{x}$ (the *reciprocal* function),
- $y = |x|$ (the *absolute value* function; recall $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$),
- $x = |y|$ (here y is **not** a function of x, since $x = 1$ allows both $y = 1$ and $y = -1$),
- $x = y^2$ (here y is **not** a function of x, for the same reason).



5. **Vertical line test:** The graph of an equation in two variables x, y represents y as a function of x if and only if every vertical line intersects it in **at most one** point. (See the above figure, where $x = |y|$ and $x = y^2$ fail this test.)

6. **Domain:** The *domain* of a function f is the set of all valid inputs $x \in \mathbb{R}$ for f, denoted D_f. So: $D_f = \{x \in \mathbb{R} \mid f(x) \in \mathbb{R} \text{ is "defined"}\}$.

   To find the domain, we exclude "problematic" values of x. The problem usually comes from:

   - division by zero;
   - square roots (or even roots, in general) of numbers < 0;
   - logarithms of numbers ≤ 0.

   Example. Let's find the domain of:

   - $f(x) = \sqrt{1 - \sqrt{2 - x}}$.
     Solution. We need $2 - x \geq 0 \implies x \leq 2$.
     Also, we need $1 - \sqrt{2 - x} \geq 0 \implies \sqrt{2 - x} \leq 1 \implies x \geq 1$.
     So $D_f = [1, 2]$. ■
   - $f(x) = \sqrt{1 - \dfrac{2}{x^2 - x}}$.
     Solution. We need $x^2 - x \neq 0 \implies x \neq 0, 1$.
     Also, we need
     $$1 - \frac{2}{x^2 - x} = \frac{x^2 - x - 2}{x(x-1)} = \frac{(x+1)(x-2)}{x(x-1)} \geq 0.$$

A sign chart gives $D_f = (-\infty, -1] \cup (0, 1) \cup [2, \infty)$. ■

Exercise. Find the domain of:

- $f(x) = \sqrt{\dfrac{x-1}{(\sqrt{x}-2)\sqrt{3-x}}}$.
- $f(x) = \dfrac{1}{2 - \sqrt{\dfrac{x^2-1}{x+5}}}$.

7. **Range.** The range of a function f is the set of all possible outputs $y \in \mathbb{R}$ of f, denoted R_f. So: $R_f = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in \mathbb{R}\}$.

   Example. Let's find the range of $f(x) = |1 - |1 - x||$.

   Solution. Note that $|1 - x| \in [0, \infty)$, so $1 - |1 - x| \in (-\infty, 1]$. In particular, $1 - |1 - x|$ can take all values in $(-\infty, 1]$. Therefore $f(x) = |1 - |1 - x|| \in [0, \infty)$, so $R_f = [0, \infty)$. ■

   Note. With domain and range specified, we can refine our notation. Instead of the generic $f : \mathbb{R} \to \mathbb{R}$, we may write $f : D_f \to R_f$ to emphasize that f has domain D_f and range R_f. (Though we will rarely do that!)

2 Composition and inverse functions

1. Composition. For $f, g : \mathbb{R} \to \mathbb{R}$:

$$(f \circ g)(x) = f(g(x)), \quad (g \circ f)(x) = g(f(x)).$$

So the order matters!

$$(f \circ f)(x) = f(f(x)), \quad (f \circ g \circ h)(x) = f(g(h(x))).$$

Example. Let $f(x) = (1-x)^2$ and $g(x) = \frac{1}{1+x}$.

- Find $(f \circ g)(2)$.
  Solution. $(f \circ g)(2) = f(g(2)) = f(1/3) = 4/9$. ■
- Find $(g \circ f \circ f)(1+\sqrt{2})$.
  Solution. $(g \circ f \circ f)(1+\sqrt{2}) = g(f(f(1+\sqrt{2}))) = g(f(2)) = g(1) = 1/2$. ■

Exercise. Let $f(x) = (1-x)^{2025}$. Find $\underbrace{(f \circ f \circ \cdots \circ f)}_{2025}(1)$.

2. Domain/Range of Compositions. To find $D_{f \circ g}$:

1. Find the expression of $f \circ g$, then find the domain of that expression.
2. Find D_g (i.e., domain of the **inner** function).
3. Exclude "problematic" inputs of g from the domain found in Step 1.

To find $R_{f \circ g}$: first find the expression of $f \circ g$, then find $D_{f \circ g}$, and then analyze.

Example. Let $f(x) = \sqrt{\frac{1}{x-1}}$ and $g(x) = \frac{1}{x-2}$. Find the expression and the domain of $f \circ g$.
Solution.

$$(f \circ g)(x) = \sqrt{\tfrac{x-2}{3-x}}.$$

Domain from expression (using a sign chart): $[2,3)$. But $D_g = \mathbb{R} - \{2\}$, so remove $x = 2$, which is problematic for g:

$$D_{f \circ g} = (2,3). \quad ■$$

Example. $f(x) = \sqrt{2 - \sqrt{3-x}}$, $g(x) = 1 - x^2$. Find the expression and the range of $(g \circ f)$.

Solution.

$$(g \circ f)(x) = 1 - (\sqrt{2 - \sqrt{3-x}})^2 = \sqrt{3-x} - 1.$$

Domain from expression: $(-\infty, 3]$. Also, we have $D_f = [-1, 3]$, so remove $(-\infty, -1)$, which is problematic for f:

$$D_{g \circ f} = [-1, 3].$$

For the range: since $D_{g \circ f} = [-1, 3]$, we have $-1 \le x \le 3$, and so $-1 \le \sqrt{3-x} - 1 \le 1$. Thus:

$$R_{g \circ f} = [-1, 1]. \quad ■$$

3. **Inverse Function.** The inverse of f, denoted f^{-1}, is the function that operates in the reverse direction of f:

$$y = f(x) \iff x = f^{-1}(y).$$

To find f^{-1}:

1. Solve the equation $y = f(x)$ for x as the unknown.
2. Swap x and y.

Example. Let $f(x) = 1 - \frac{2}{3-\frac{4}{5-x}}$.

- Find $f^{-1}(x)$.
  Solution.

$$y = 1 - \frac{2}{3 - \frac{4}{5-x}} = \frac{1-x}{11-3x}.$$

  Solve $y = \frac{1-x}{11-3x}$ for x. So we have $x = \frac{11y-1}{3y-1}$. Now, swap x and y:

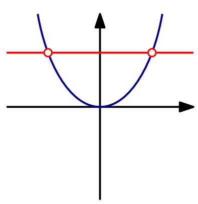
$$f^{-1}(x) = \frac{11x-1}{3x-1}. \quad \blacksquare$$

- Compute $(f^{-1} \circ f^{-1})(-1)$.
  Solution. $(f^{-1} \circ f^{-1})(-1) = f^{-1}(f^{-1}(-1)) = f^{-1}(3) = 4.$ $\blacksquare$

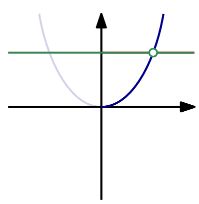
4. **Horizontal Line Test.** A function f has an inverse $\iff$ every horizontal line hits the graph of f **at most once**.

Example. The function $f(x) = x^2$ fails (since $f(2) = f(-2) = 4$).

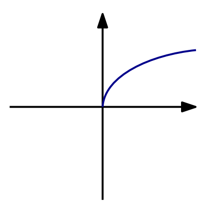
One way to fix this is to restrict the domain of f. For example, if we set $D_f = [0, \infty)$, then $f^{-1}(x) = \sqrt{x}$ exists.



$y = x^2$



$y = x^2$
$x \geq 0$



$y = \sqrt{x}$

5. **Properties of Inverse.** If f has an inverse f^{-1}, then:

- $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.
- The graph of f^{-1} is the reflection of the graph of f across the line $y = x$.
- $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$.

Example. Let $f(x) = \frac{1-x^3}{1+x^3}$. Find the expression, the domain, and the range of f^{-1}.

Solution.

$$y = \tfrac{1-x^3}{1+x^3} \;\Rightarrow\; y(1+x^3) = 1 - x^3 \;\Rightarrow\; x = \sqrt[3]{\tfrac{y-1}{y+1}} \;\Rightarrow\; f^{-1}(x) = \sqrt[3]{\tfrac{x-1}{x+1}}.$$

Problem for f at $x = -1$. Thus $D_f = \mathbb{R} - \{-1\}$, and therefore $D_{f^{-1}} = R_f = \mathbb{R} - \{-1\}$. Likewise, $R_{f^{-1}} = D_f = \mathbb{R} - \{-1\}$. $\blacksquare$

3 Transformations, even and odd functions

1. **Transformations.** We can transform the graph of a function $y = f(x)$ to obtain graphs of related functions.

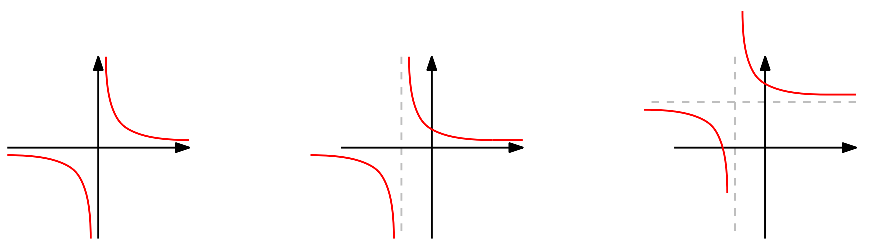
Shifts: For $a \in \mathbb{R}$, $a \neq 0$:

1. **Horizontal**: $y = f(x - a)$.
   - If $a > 0$, this is a shift by a units *to the right.*
   - If $a < 0$, shift by $|a|$ units *to the left.*
2. **Vertical**: $y = f(x) + a$.
   - If $a > 0$, shift *up* by a units.
   - If $a < 0$, shift *down* by $|a|$ units.

   ▶ Note that $y = f(x) + a$ is the same as $y - a = f(x)$, which conceptually matches the horizontal case.

Example. Find the graph of $f(x) = \dfrac{3x + 4}{x + 1}$.

Solution. We have $f(x) = \frac{3x+4}{x+1} = \frac{1}{x-(-1)} + 3$. So start with $y = \dfrac{1}{x}$, and:



$y = \frac{1}{x}$ $\qquad$ $y = \frac{1}{x-(-1)}$ $\qquad$ $y = \frac{1}{x-(-1)} + 3$

■

Stretch, Compression, and Reflection. For some $k \in \mathbb{R}$, $k \neq 0$:

1. **Horizontal**: $y = f(kx)$.
   - If $|k| > 1$, then a horizontal compression, **and** a reflection about the y-axis **if** $k < 0$.
   - If $|k| < 1$, then a horizontal stretch, **and** a reflection about the y-axis **if** $k < 0$.
2. **Vertical**: $y = kf(x)$.
   - If $|k| > 1$, then a vertical stretch, and a reflection about the x-axis if $k < 0$.
   - If $|k| < 1$, then a vertical compression, **and** a reflection about the x-axis **if** $k < 0$.

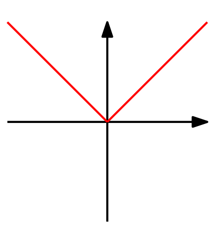
   ▶ Note that $y = kf(x)$ is the same as $\dfrac{y}{k} = f(x)$, which conceptually matches the horizontal case.

We can also combine the transformations. Here is the right order to apply them:

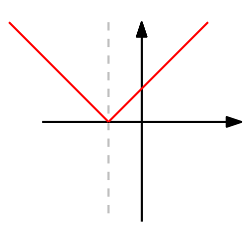
1. Horizontal shift.
2. Horizontal stretch/compression/reflection.
3. Vertical stretch/compression/reflection.
4. Vertical shift.

Example. Find the graph of $f(x) = 1 - |1 - 2x|$.

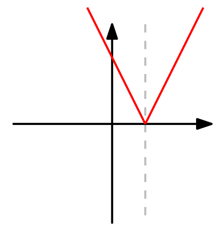
Solution. We have $f(x) = -|-2x - (-1)| + 1$. Start with the graph of $y = |x|$, and:



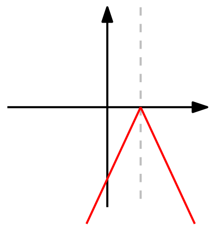
$y = |x|$



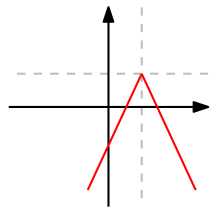
$y = |x - (-1)|$



$y = |-2x - (-1)|$



$y = -|-2x - (-1)|$



$y = -|-2x - (-1)| + 1$

■

Exercise. Find the graph of $f(x) = \dfrac{2x-1}{1-3x}$.

2. Even and odd functions. A function f is said to be

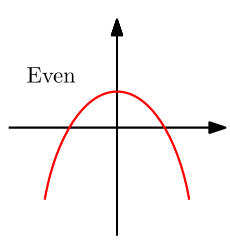
- *even* if $f(-x) = f(x)$ for all $x \in D_f$,
- *odd* if $f(-x) = -f(x)$ for all $x \in D_f$.

Example.

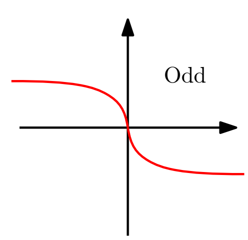
- $f(x) = 1 - x^2$ is even because $f(-x) = 1 - (-x)^2 = 1 - x^2 = f(x)$.
- $f(x) = -\sqrt[3]{x}$ is odd because $f(-x) = -\sqrt[3]{-x} = \sqrt[3]{x} = -f(x)$.
- $f(x) = \dfrac{x}{1-x}$ is neither odd nor even, because $f(2) = -2$ but $f(-2) = -\dfrac{2}{3}$.

Graphs of even and odd functions have special symmetries:

- If f is even, its graph is symmetric across the y-axis.
- If f is odd, its graph is symmetric across both axes; and so it is symmetric across the origin.



$y = 1 - x^2$



$y = -\sqrt[3]{x}$

3. Even–odd decomposition. For every function f, the *even part of* f is defined as

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

and the *odd part of* f is defined as

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Properties of the even and the odd part:

- f_e is even.

  Proof: $f_e(-x) = \dfrac{f(-x) + f(x)}{2} = \dfrac{f(x) + f(-x)}{2} = f_e(x)$. ■
- f_o is odd.

  Proof: $f_o(-x) = \dfrac{f(-x) - f(x)}{2} = -\dfrac{f(x) - f(-x)}{2} = -f_o(x)$. ■
- $f(x) = f_e(x) + f_o(x)$ for all $x \in D_f$.

  Proof: Trivial. ■

So every function is the sum of an even function, namely the even part f_e, and an odd function, namely the odd part f_o.

Example. Find the even and the odd part of $f(x) = \dfrac{x}{1-x}$.

Solution.

$$f_e(x) = \frac{f(x) + f(-x)}{2} = \frac{1}{2}\left(\frac{x}{1-x} - \frac{x}{1+x}\right) = \frac{x^2}{1-x^2},$$

$$f_o(x) = \frac{f(x) - f(-x)}{2} = \frac{1}{2}\left(\frac{x}{1-x} + \frac{x}{1+x}\right) = \frac{x}{1-x^2}. \quad ■$$

Example. Prove that if f is an odd function, then $g(x) = x - f(x)$ is also odd.

Solution. $g(-x) = -x - f(-x) = -x + f(x) = -(x - f(x)) = -g(x)$. ■

Example. Prove that if f and g are odd, then $f \circ g$ is also odd.

Solution. $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$. ■

Example. Show that the only function that is both even and odd is $f(x) = 0$ for all x.

Solution. If f is both even and odd, then for all $x \in D_f$ we have

$$f(-x) = f(x) \quad \text{and} \quad f(-x) = -f(x).$$

Hence $f(x) = -f(x)$, so $2f(x) = 0$ and therefore $f(x) = 0$ for all x in its domain. ■

Exercise. Let f be odd and let $h(x) = x - f\big(x + f(x - f(x))\big)$. Is h odd, even, or neither?

Exercise. Prove that the even and the odd part of every function is unique. That is, for every function f, if $f = g_1 + h_1 = g_2 + h_2$ such that g_1, g_2 are even and h_1, h_2 are odd, then $g_1 = g_2$ and $h_1 = h_2$.

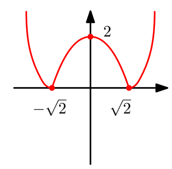
4 Absolute value, piecewise and heaviside functions

1. Absolute Value. Basic example of a piecewise-defined function: $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$

A function f is **piecewise-defined** if we partition the real line into intervals, and then give each interval its own expression of f.

Example. Write $f(x) = |2 - x^2|$ in piecewise form, and sketch the graph.
Solution. The sign of $2 - x^2$ changes at $x = \pm\sqrt{2}$. So by a sign chart:

$$f(x) = |2{-}x^2| = \begin{cases} 2 - x^2 & -\sqrt{2} \leq x \leq \sqrt{2}, \\ x^2 - 2 & x < -\sqrt{2} \text{ or } x > \sqrt{2} \end{cases}$$



2. Properties of absolute value.

1. We have $|x| = |-x|$ for all $x \in \mathbb{R}$. In particular $f(x) = |x|$ is even.
2. Works "as expected" with multiplication and division:
$$|xy| = |x||y|, \qquad \left|\frac{x}{y}\right| = \frac{|x|}{|y|},\ y \neq 0.$$
3. Does **not** work "as expected" with addition: $|2 + (-3)| = 1 \neq 5 = |2| + |-3| = 5$.
But we have the **triangle inequality**: $|x + y| \leq |x| + |y|$, for all $x, y \in \mathbb{R}$.
Exercise. Prove that $|x| - |y| \leq |x - y|$.
4. Recall that by $\sqrt{\cdot}$ we always mean the *positive* root. So $\sqrt{x^2} = |x|$.
5. For $x, a \in \mathbb{R}$, $|x - a|$ is the distance between the two points x and a on the real line. So
$$|x - a| < \delta \iff a - \delta < x < a + \delta.$$
$$|x - a| > \delta \iff x < a - \delta \quad \text{or} \quad x > a + \delta.$$
This distance interpretation is often useful, especially when solving inequalities.

Example. Solve $|3x + 1| \leq 5$.
Solution. $|3x - (-1)| \ \Rightarrow\ (-1) - 5 \leq 3x \leq (-1) + 5 \ \Rightarrow\ -6 \leq 3x \leq 4 \ \Rightarrow\ x \in [-2, \frac{4}{3}]$. ■

Example. Solve $\left|\dfrac{6}{7x - 4}\right| \leq 3$.

Solution. $\left|\dfrac{6}{7x - 4}\right| \leq 3 \ \Rightarrow\ |7x - 4| \geq 2 \ \Rightarrow\ 7x \leq 4 - 2$ or $7x \geq 4 + 2 \ \Rightarrow\ 7x \leq 2$ or $7x \geq 6 \Rightarrow\ x \in (-\infty, \frac{2}{7}] \cup [\frac{6}{7}, \infty)$. ■

Example. Solve $\left|\dfrac{3 - 2x}{x - 1}\right| > 2$.

Solution. This means $|3/2 - x| > |x - 1|$. So x is closer to 1 than to $3/2$. Hence $x \in (-\infty, \frac{5}{4})$. We also need to take out $x = 1$ because it is the root of the denominator. So the answer is $x \in (-\infty, 1) \cup (1, \frac{5}{4})$.

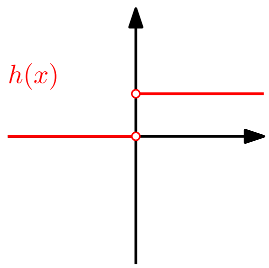
3. Heaviside (Step) Function. It is a piecewise-defined function with the following expression (in particular, undefined at $x = 0$).

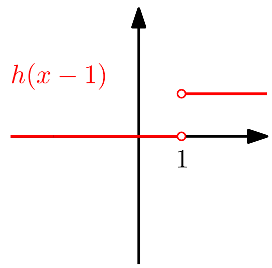
$$h(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0, \end{cases}$$

More generally, for $a \in \mathbb{R}$,

$$h(x-a) = \begin{cases} 0 & x < a, \\ 1 & x > a, \end{cases}$$

is like a "switch" that is off for $x < a$ and on for $x > a$.





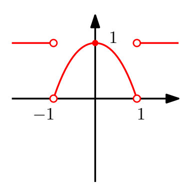
Example. Write $f(x) = 1 - x^2\, h\left(\dfrac{1-x}{1+x}\right)$ in piecewise form, and sketch its graph.

Solution. Note that $h\left(\dfrac{1-x}{1+x}\right) = 0$ when $\dfrac{1-x}{1+x} < 0$ and $h\left(\dfrac{1-x}{1+x}\right) = 1$ when $\dfrac{1-x}{1+x} > 0$. Also, by a sign chart, we have

$$\frac{1-x}{1+x} < 0 \iff x < -1 \text{ or } x > 1$$

$$\frac{1-x}{1+x} > 0 \iff -1 < x < 1.$$

So, $f(x) = \begin{cases} 1 & x < -1 \text{ or } x > 1 \\ 1 - x^2 & -1 < x < 1 \end{cases}$. ■



Key observation. We can:

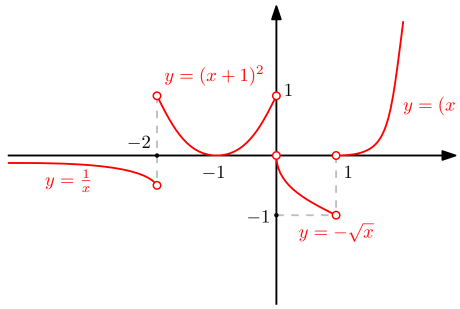
-*write every piecewise-defined function in terms of Heavisides of the form $h(x-a)$;*

-*write expressions involving Heavisides of the form $h(x-a)$ as a piecewise-defined function.*

Algorithm.

1. Find the transition points (roots inside h) and line them up in order.
2. The "leftmost function" (valid down to $-\infty$) has no Heaviside next to it.
3. Moving rightward: at each transition point a:
   – subtracting $f(x)h(x-a)$ turns off a function $f(x)$ for $x > a$.
   – adding $f(x)h(x-a)$ turns on a function $f(x)$ for $x > a$.

Example. Write the function with the following graph as a Heaviside sum.



Solution. The transition points are $-2, 0, 1$. We have

$$f(x) = \underbrace{\frac{1}{x}}_{\text{Left most}} \underbrace{-\frac{1}{x}h(x+2) + (x+1)^2 h(x+2)}_{\text{At } x=-2\text{, turn off } \frac{1}{x} \text{ and turn on } (x+1)^2} \underbrace{-(x+1)^2 h(x) - \sqrt{x}h(x)}_{\text{At } x=0\text{, turn off } (x+1)^2 \text{ and turn on } -\sqrt{x}} \underbrace{+\sqrt{x}h(x-1) + (x-1)^3 h(x-1)}_{\text{At } x=1\text{, turn off } -\sqrt{x} \text{ and turn on } (x-1)^3}. \quad \blacksquare$$

Example. Write the following piecewise-defined function as a Heaviside sum.

$$f(x) = \begin{cases} 1 & x < -1 \\ -x & -1 < x < 1, \\ x^2 & 1 < x < \frac{3}{2}, \\ -x^3 & x > \frac{3}{2}, \end{cases}$$

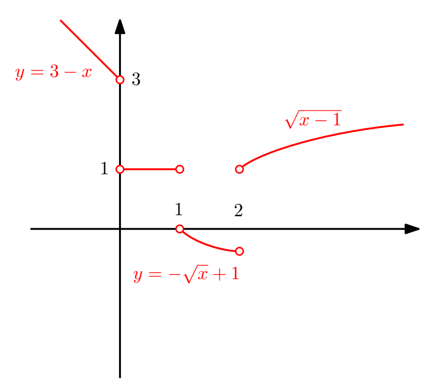
Solution. We have $f(x) = 1 + (-1-x)h(x+1) + (x+x^2)h(x-1) + (-x^2-x^3)h(x-\frac{3}{2})$. ■

Example. Write this Heaviside sum as a piecewise-defined function and sketch its graph.

$$f(x) = 3 - x + (x-2)h(x) - \sqrt{x}h(x-1) + (\sqrt{x-1} + \sqrt{x} - 1)h(x-2)$$

Solution. The transition points are $0, 1, 2$. We have

$$f(x) = \begin{cases} 3 - x & x < 0 \\ 1 & 0 < x < 1, \\ -\sqrt{x} + 1 & 1 < x < 2, \\ \sqrt{x-1} & x > 2, \end{cases}$$



■

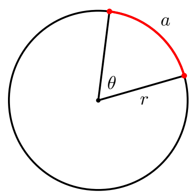
5 Trigonometry review

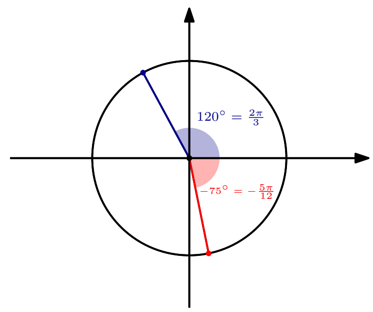
1. **Angles.** We measure angles in *radians*. Important to know:

   - In a circle of radius r, the angle in front of an arc of length a is $\frac{a}{r}$ radians. A full circle ($360°$) is 2π radians. Conversion rules:

$$\text{radians} = \frac{\pi}{180}\ \text{degrees}, \quad \text{degrees} = \frac{180}{\pi}\ \text{radians}.$$

   - We allow all real numbers to be angles, using the xy-plane. Start from the positive x-axis; positive angles are measured counterclockwise, negative angles clockwise.





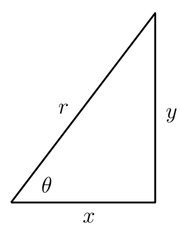
2. **Trigonometric functions.** In a right triangle:

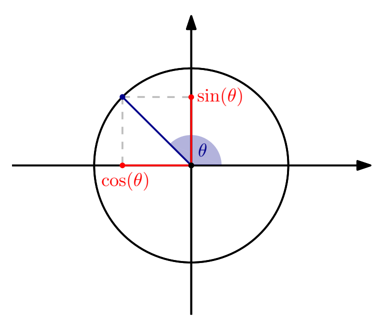
   - Main functions: $\sin(\theta) = \frac{y}{r}$, $\cos(\theta) = \frac{x}{r}$, $\tan(\theta) = \frac{y}{x}$.

   - Reciprocals: $\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{r}{y}$, $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{r}{x}$, $\cot(\theta) = \frac{1}{\tan\theta} = \frac{x}{y}$.

   For a general angle θ,

   1. Find the point on the unit circle corresponding to θ.
   2. The y-coordinate is $\sin(\theta)$, the x-coordinate is $\cos(\theta)$.





3. **Basic identities.** (You should memorize these.)

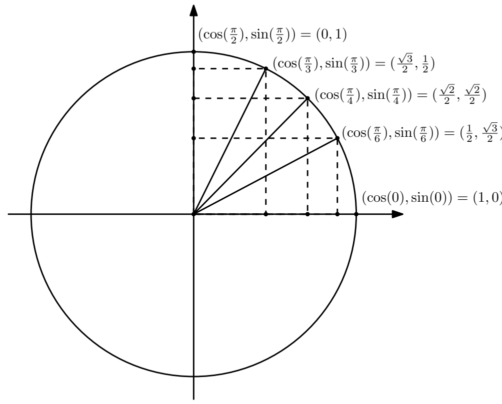
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}, \quad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)},$$

$$\sin^2(\theta) + \cos^2(\theta) = 1,$$

$$1 + \tan^2(\theta) = \frac{1}{\cos^2(\theta)} = \sec^2(\theta),$$

$$1 + \cot^2(\theta) = \frac{1}{\sin^2(\theta)} = \csc^2(\theta).$$

4. Values to memorize.



For $\theta \in \{0, \ \pi/6, \ \pi/4, \ \pi/3, \ \pi/2\}$ (meaning $0°, 30°, 45°, 60°, 90°$) memorize all six trigonometric functions. These are all in the first quadrant. Also, memorize trigonometric values of $\pi/2 + \theta$ (2nd quadrant), $\pi + \theta$ (3rd quadrant), $3\pi/2 + \theta$ (4th quadrant).
Or, you can use the following algorithm. This in fact works for *any* θ (just pretend at the start that θ is in quadrant 1).

1. Figure out which quadrant you are in.
2. Note the sign of x (cos) and y (sin) in that quadrant.
3. In quadrants 2 and 4, sin and cos swap. In quadrants 1 and 3, they remain the same.

Example. $\cos(\frac{3\pi}{4}) = \cos(\frac{\pi}{2} + \frac{\pi}{4})$. This is in quadrant 2: cos is negative and $\sin, \cos$ should be swapped. So $\cos(\frac{3\pi}{4}) = -\sin(\frac{\pi}{4}) = -\frac{\sqrt{2}}{2}$. ■

Example. $\tan(\frac{7\pi}{6}) = \tan(\pi + \frac{\pi}{6})$. This is in quadrant 3: both $\sin, \cos$ are negative, so tan is positive. Also, no swap needed. So $\tan(\frac{7\pi}{6}) = \tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. ■

Exercise. Using the above rules, find formulas for trigonometric functions of $\frac{\pi}{2} + \theta$, $\pi + \theta$, and $\frac{3\pi}{2} + \theta$ for any θ.

5. Other important identities. (You should memorize these too!)

- Sums and differences:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

- Double angles:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta).$$

Example. Prove that $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$.

Solution. We have $\tan(\alpha \pm \beta) = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} = \frac{\sin\alpha\cos\beta\pm\cos\alpha\sin\beta}{\cos\alpha\cos\beta\mp\sin\alpha\sin\beta}$.
Divide numerator and denominator by $\cos\alpha\cos\beta$:

$$\tan(\alpha \pm \beta) = \frac{\frac{\sin\alpha}{\cos\alpha} \pm \frac{\sin\beta}{\cos\beta}}{1 \mp \frac{\sin\alpha}{\cos\alpha}\frac{\sin\beta}{\cos\beta}} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\,\tan\beta}. \quad ■$$

Example. Show that $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2\theta}$.

Solution. Use the above example with $\alpha = \beta = \theta$. ■

Example. Show that $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$.

Solution. Using the sum identity, we have $\cos(3\theta) = \cos(2\theta+\theta) = \cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)$. So by the double angle identities, we have

$$\cos(3\theta) = (\cos^2(\theta) - \sin^2(\theta))\cos(\theta) - (2\sin(\theta)\cos(\theta))\sin(\theta) = \cos^3(\theta) - 3\sin^2(\theta)\cos(\theta).$$

So: $\cos(3\theta) = \cos^3(\theta) - 3(1 - \cos^2(\theta))\cos(\theta) = 4\cos^3(\theta) - 3\cos(\theta)$. ■

Example.

(a) Show that $\tan(\dfrac{\theta}{2}) = \dfrac{\sin(\theta)}{1+\cos(\theta)}$.

Solution. Using double angle identities, we have:

$$\frac{\sin(\theta)}{1+\cos(\theta)} = \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{1 + \big(2\cos^2(\frac{\theta}{2}) - 1\big)} = \frac{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})}{2\cos^2(\frac{\theta}{2})} = \tan(\tfrac{\theta}{2}). \quad ■$$

(b) Find all $\theta \in [0, 2\pi]$ such that $\cot\theta = \tan(\frac{\theta}{2})$.

Solution. Using part (a), for angles where both sides are defined we have

$$\cot\theta = \tan(\tfrac{\theta}{2}) \;\Rightarrow\; \frac{\cos(\theta)}{\sin(\theta)} = \frac{\sin(\theta)}{1+\cos(\theta)} \;\Rightarrow\; \cos(\theta) + \cos^2(\theta) = \sin^2(\theta) = 1 - \cos^2(\theta).$$

And so

$$\cos(\theta) + \cos^2(\theta) = 1 - \cos^2(\theta) \;\Rightarrow\; 2\cos^2(\theta) + \cos(\theta) - 1 = 0 \;\Rightarrow\; (2\cos(\theta) - 1)(\cos(\theta) + 1) = 0.$$

Thus $\cos(\theta) = \frac{1}{2}$ or $\cos(\theta) = -1$. On $[0, 2\pi]$ this gives

$$\cos(\theta) = \tfrac{1}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}, \frac{5\pi}{3}; \qquad \cos(\theta) = -1 \;\Rightarrow\; \theta = \pi.$$

We must enforce the domain: $\cot(\theta)$ is undefined when $\sin(\theta) = 0$ and $\tan(\frac{\theta}{2})$ is undefined when θ is an odd multiple of π. Hence $\theta = \pi$ is excluded. Therefore $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$ are the solutions. ■

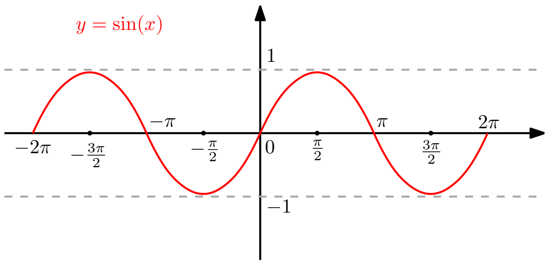
Exercise. Find $\cot(\alpha + \beta)$ and $\cot(2\theta)$ only in terms of cot.

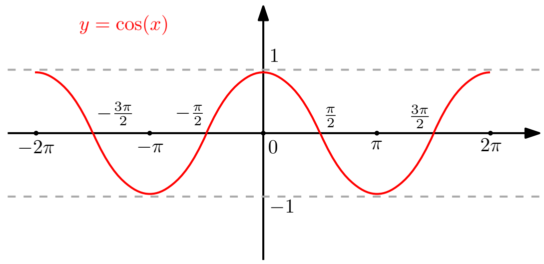
Exercise. Prove that $\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

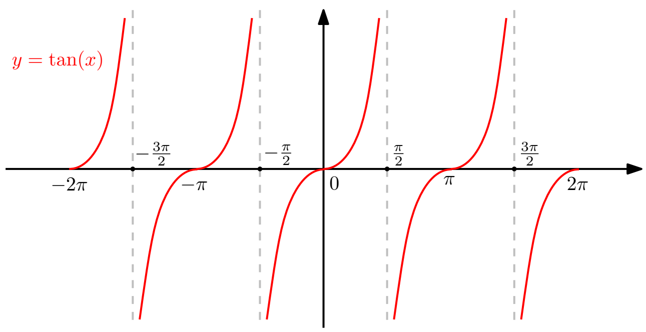
Exercise. Let $\alpha + \beta + \gamma = \pi$. Show that $\tan(\alpha) + \tan(\beta) + \tan(\gamma) = \tan(\alpha)\tan(\beta)\tan(\gamma)$.

6 Inverse of trigonometric functions and generalized sine functions

1. Graphs of trigonometric functions. Here they are:







Properties:

- $y = \sin(x)$: domain $\mathbb{R}$, range $[-1, 1]$, periodic with period 2π, because $\sin(x + 2\pi) = \sin(x)$.
- $y = \cos(x)$: domain $\mathbb{R}$, range $[-1, 1]$, periodic with period 2π, because $\cos(x + 2\pi) = \cos(x)$.
- Graphs of $y = \sin(x)$ and $y = \cos(x)$ are horizontal shifts of each other by $\pi/2$ because:

$$\sin(x) = \cos\left(x - \frac{\pi}{2}\right), \ \cos(x) = \sin\left(x + \frac{\pi}{2}\right).$$

- $y = \tan(x)$: domain $\mathbb{R} - \{\frac{\text{odd integer times } \pi}{2}\}$, range $\mathbb{R}$, periodic with period π.

2. Inverses. The functions $y = \sin(\theta)$, $y = \cos(\theta)$, $y = \tan(\theta)$ are **not** invertible on all of $\mathbb{R}$ since they fail the horizontal line test.

But, we restrict their domains to obtain invertible functions:

- $\sin(\theta)$ restricted to $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ gives:

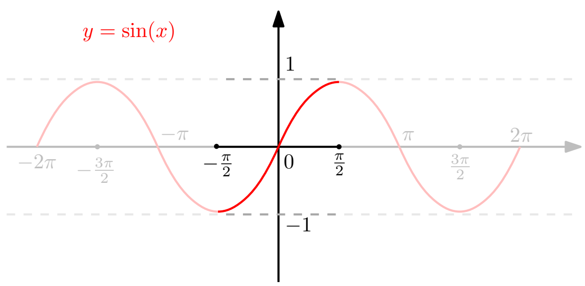
$$\sin^{-1} : [-1, 1] \to [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]. \text{ (also written } \arcsin)$$

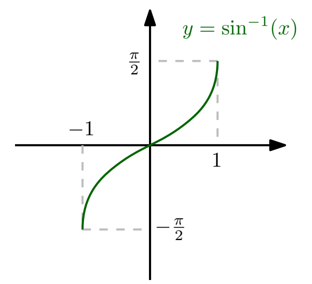
- $\cos(\theta)$ restricted to $\theta \in [0, \pi]$ gives

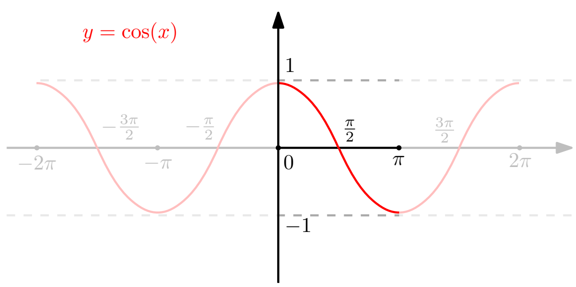
$$\cos^{-1} : [-1, 1] \to [0, \pi]. \text{ (also written } \arccos)$$

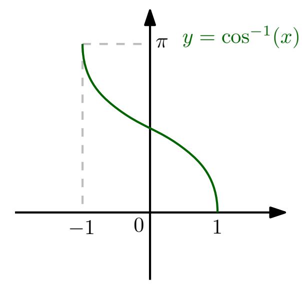
- $\tan(\theta)$ restricted to $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ gives

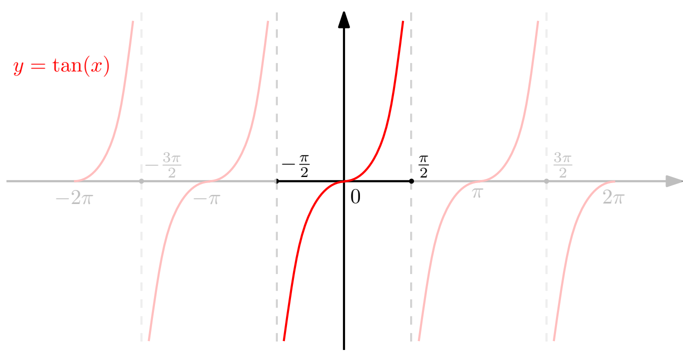
$$\tan^{-1} : \mathbb{R} \to (-\tfrac{\pi}{2}, \tfrac{\pi}{2}). \text{ (also written } \arctan)$$

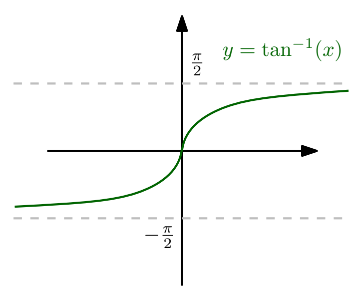












The functions $\csc^{-1}, \sec^{-1}, \cot^{-1}$ are also defined accordingly.

Example. Compute $\csc^{-1}(-\sqrt{2})$.

Solution. Let $\theta = \csc^{-1}(-\sqrt{2})$. Then $\csc(\theta) = -\sqrt{2} \Rightarrow \sin(\theta) = \frac{1}{-\sqrt{2}} = \frac{-\sqrt{2}}{2} \Rightarrow \theta = \sin^{-1}(\frac{-\sqrt{2}}{2})$. Since $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, it follows that θ is in the fourth quadrant, and so $\theta = -\frac{\pi}{4}$. ■

Example. Compute $\cot\big(\cos^{-1}(1/100)\big)$.

Solution. Let $\theta = \cos^{-1}(\frac{1}{100})$. Then $\cos(\theta) = \frac{1}{100}$ with $\theta \in [0, \pi]$. So $\tan^2(\theta) = \frac{1}{\cos^2(\theta)} - 1 = 9999$. Since $\cos(\theta) > 0$ and $\theta \in [0, \pi]$, it follows that $\theta \in [0, \frac{\pi}{2}]$, and so $\tan(\theta) > 0$. Thus, $\tan(\theta) = \sqrt{9999} \Rightarrow \cot(\theta) = \frac{1}{\sqrt{9999}}$. ■

Example. Compute $\cot\big(2\sin^{-1}(\frac{1}{3})\big)$.

Solution. Let $\theta = \sin^{-1}\big(\frac{1}{3}\big)$. Then $\sin(\theta) = \frac{1}{3}$, with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. So $\cos^2(\theta) = 1 - \sin^2(\theta) = \frac{8}{9}$. Since $\sin(\theta) > 0$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we have $\theta \in [0, \frac{\pi}{2}]$, and so $\cos(\theta) > 0$. Thus $\cos(\theta) = \frac{2\sqrt{2}}{3}$.

Now, by the double angle identities, we have:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{4\sqrt{2}}{9}; \quad \cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \frac{7}{9}.$$

Therefore, $\cot(2\theta) = \frac{\cos(2\theta)}{\sin(2\theta)} = \frac{7/9}{4\sqrt{2}/9} = \frac{7}{4\sqrt{2}} = \frac{7\sqrt{2}}{8}$. ■

Exercise. Compute $\cot\left(\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)\right)$.

3. **Generalized sine functions.** A function of the form

$$f(x) = A\sin(\omega x + \varphi),$$

where A, ω, φ are constants. We call ω the *angular frequency* (motivated by physics stuff).

Although it may not appear so at first, every function of the form $f(x) = B\sin(\omega x) + C\cos(\omega x)$ is a generalized sine function. Indeed, we can write

$$B\sin(\omega x) + C\cos(\omega x) = A\sin(\omega x + \varphi) = A\cos(\varphi)\sin(\omega x) + A\sin(\varphi)\cos(\omega x)$$

and so $B = A\cos(\varphi)$ and $C = A\sin(\varphi)$. We deduce that

$$B^2 + C^2 = A^2(\cos^2(\varphi) + \sin^2(\varphi)) = A^2 \Rightarrow A = \sqrt{B^2 + C^2}.$$

Note that we *choose* $A > 0$ by convention. Also, we get:

$$\cos(\varphi) = \frac{B}{A}, \quad \sin(\varphi) = \frac{C}{A}.$$

You are free to *choose* any φ that satisfies the above two equations. Just make sure that φ is chosen in the correct quadrant according to the signs of *both* B and C.

Example. Find all x such that $\sqrt{3}\cos(11x) - \sin(11x) = 1$.

Solution. First, we write $\sqrt{3}\cos(11x) - \sin(11x)$ in the form $A\sin(\omega x + \varphi)$. We have $B = -1$ and $C = \sqrt{3}$. So $A = \sqrt{B^2 + C^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$.

Also, $\cos(\varphi) = \frac{B}{A} = \frac{-1}{2}$ and $\sin(\varphi) = \frac{C}{A} = \frac{\sqrt{3}}{2}$. Thus, φ is in quadrant 2, and for example we may choose $\varphi = \frac{2\pi}{3}$. Therefore $\sqrt{3}\cos(11x) - \sin(11x) = 2\sin(11x + \frac{2\pi}{3})$.

Now, we want to find all x such that $2\sin(11x + \frac{2\pi}{3}) = 1$. So $\sin(11x + \frac{2\pi}{3}) = \frac{1}{2}$ and hence

$$11x + \tfrac{2\pi}{3} = 2k\pi + \tfrac{\pi}{6} \quad \text{or} \quad 11x + \tfrac{2\pi}{3} = 2k\pi + \tfrac{5\pi}{6} \quad \text{for } k \in \mathbb{Z}.$$

Thus

$$11x = -\tfrac{\pi}{2} + 2k\pi \quad \text{or} \quad 11x = \tfrac{\pi}{6} + 2k\pi,$$

so

$$x = -\tfrac{\pi}{22} + \tfrac{2k\pi}{11} \quad \text{or} \quad x = \tfrac{\pi}{66} + \tfrac{2k\pi}{11}, \qquad k \in \mathbb{Z}. \quad ■$$

Exercise. Sketch the graph of $f(x) = \sin(\frac{x-\pi}{2}) - \cos(\frac{x-\pi}{2})$.

7 Exponential, logarithmic, and hyperbolic trigonometric functions

1. **Exponential and logarithmic functions.** For $a > 0$ and $a \neq 1$, the *exponential* function with base a is $f(x) = a^x$. It has domain $\mathbb{R}$ and its range is $(0, \infty)$.

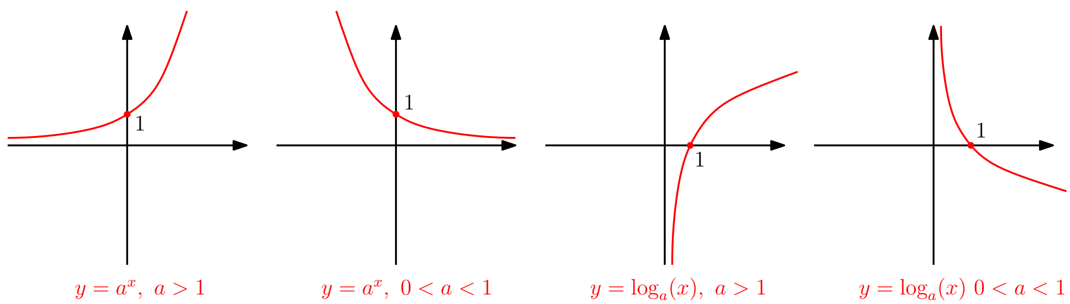
   The *logarithmic* function in base a is the inverse of the exponential with base a:

   $$y = \log_a(x) \quad \Longleftrightarrow \quad a^y = x.$$

   Its domain is $(0, \infty)$ and its range is $\mathbb{R}$.

   For example, $\log_5(125) = 3$ because $5^3 = 125$ and $\log_{27}\left(\frac{1}{3}\right) = -\frac{1}{3}$ because $27^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}$.

   The graphs look different for $a > 1$ and for $0 < a < 1$.



   Laws of exponents and logarithms (you need to memorize these):

   1. Exponential and logarithm are inverses: $a^{\log_a(b)} = b$.
   2. $a^{b+c} = a^b \cdot a^c$, and $\log_a(bc) = \log_a(b) + \log_a(c)$.
   3. $a^{b-c} = \frac{a^b}{a^c}$, and $\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c)$. In particular, $a^{-c} = \frac{1}{a^c}$ and $\log_a\left(\frac{1}{a}\right) = -1$.
   4. $(a^b)^c = a^{bc}$, and $c \log_a(b) = \log_a(b^c)$.

   There is a special base, the *Euler constant*: $e \approx 2.718\ldots$. This is an irrational number that appears in many applications. It can be characterized in several equivalent ways. For instance:

   - e is the unique base a such that the slope of the graph of $y = a^x$ at $x = 1$ is exactly 1 (more on this when we get to derivatives).
   - $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.
   - $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$.

   In this base, the logarithm has a special name and symbol: $\log_e(x) = \ln(x)$.

   Example. Solve the following equations.

   (a) $\log_4(3x + 1) = 3$.

   Solution. $3x + 1 = 4^3 = 64 \Rightarrow 3x = 63 \Rightarrow x = 21$. ■

   (b) $\log_{6\sqrt{2}}(x - 1) - \log_{6\sqrt{2}}(1/x) = 2$.

   Solution. Combine the logs: $\log_{6\sqrt{2}}\left(\frac{x-1}{1/x}\right) = \log_{6\sqrt{2}}(x^2 - x) = 2$. So $x^2 - x = (6\sqrt{2})^2 = 72$. Hence $x^2 - x - 72 = 0$, and $(x - 9)(x + 8) = 0$. So $x = 9$ and $x = -8$. But $x = -8$ is not acceptable because log has domain $(0, \infty)$. Thus, $x = 9$ is the only solution. ■

(c) $3^{\log_9(5+x-2x^2)} = 2$.

Solution. Let $t = \log_9(5 + x - 2x^2)$. Then

$$9^t = 5 + x - 2x^2 \Rightarrow 3^t = (9^{1/2})^t = 9^{t/2} = (9^t)^{1/2} = \sqrt{5 + x - 2x^2}.$$

Hence $\sqrt{5 + x - 2x^2} = 2 \;\Rightarrow\; 5 + x - 2x^2 = 4 \;\Rightarrow\; 2x^2 - x - 1 = 0 \Rightarrow (2x + 1)(x - 1) = 0$; so $x = 1$ or $x = -\frac{1}{2}$. Both values are acceptable for the domain of log. ■

(d) $\ln(1 - \ln(1 + \ln(x))) = 1$.

Solution. Exponentiate both sides: $1-\ln(1+\ln(x)) = e$. So $\ln(1+\ln(x)) = 1-e$. Exponentiate again: $1 + \ln(x) = e^{1-e}$. So $\ln(x) = e^{1-e} - 1$. Hence $x = e^{e^{1-e}-1}$. ■

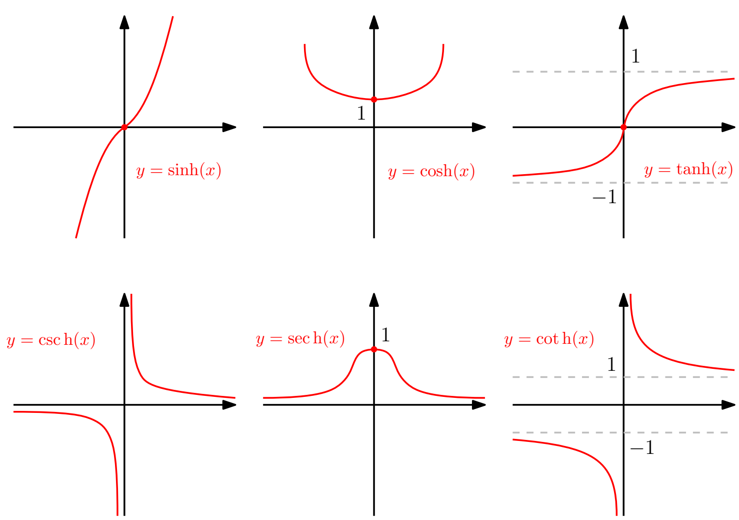
2. Hyperbolic trigonometric functions.

The *hyperbolic cosine* and *sine* are defined as the even and the odd parts of e^x. The hyperbolic tangent is also defined as expected:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Other hyperbolic trigonometric functions are defined accordingly:

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}, \quad \operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$



These are called "trigonometric" because they satisfy identities very similar to the ordinary trigonometric functions.

Example. Show that $\cosh^2(x) - \sinh^2(x) = 1$.

Solution. Note that $\cosh^2(x) - \sinh^2(x) = (\cosh(x) + \sinh(x))(\cosh(x) - \sinh(x))$. Also,

$$\cosh(x)+\sinh(x) = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = e^x; \quad \cosh(x)-\sinh(x) = \frac{e^x + e^{-x} - e^x + e^{-x}}{2} = e^{-x}.$$

So $\cosh^2(x) - \sinh^2(x) = e^x e^{-x} = e^0 = 1$. ■

Example. Show that $\sinh(2x) = 2\sinh(x)\cosh(x)$.

Solution. We have $\sinh(2x) = \frac{e^{2x}-e^{-2x}}{2}$, and

$$2\sinh(x)\cosh(x) = 2 \cdot \frac{e^x - e^{-x}}{2} \cdot \frac{e^x + e^{-x}}{2} = \frac{(e^x - e^{-x})(e^x + e^{-x})}{2} = \frac{e^{2x} - e^{-2x}}{2}. \quad \blacksquare$$

Exercise. Let $f(x) = \tanh(x)$. Find the expression, domain, and range of f^{-1}.

Exercise. Show that $\tanh(x+y) = \frac{\tanh(x)+\tanh(y)}{1+\tanh(x)\tanh(y)}$.

8 Limits

1. **Definition.** For a function f and a number $a \in \mathbb{R}$, if there is a finite number L such that $f(x)$ gets closer and closer to L as x approaches a (but never becomes a), then we say that "the limit of f as x approaches a is L, and denote it by: $\lim_{x\to a} f(x) = L$. For example, $\lim_{x\to\sqrt{2}} x^2 = 2$.

   More precisely, in order for $\lim_{x\to a} f(x)$ to exist, the following should hold:

   - $f(x)$ should approach "something" as $x \to a$ (i.e., it shouldn't oscillate).

   - $f(x)$ should not blow up to $\pm\infty$.

   - $f(x)$ must approach the same number L as x approaches a both from left and right:

   $$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

   Otherwise, we say that the limit does not exist (DNE).

   Note. There is no need for $f(a)$ to be equal to L, or even be defined.

   Example. Which of the following limits exist? Why?

   (a) $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$.

   Solution. DNE, values oscillate constantly between -1 and 1. ■

   (b) $\lim_{x\to 0} \frac{1}{x}$.

   Solution. DNE, tends to $\pm\infty$. ■

   (c) $\lim_{x\to 0} 2 - x\left|\frac{1}{x}\right|$.

   Solution. DNE. Note that for $x > 0$, we have $2 - x\left|\frac{1}{x}\right| = 2 - x \cdot \frac{1}{x} = 1$, and for $x < 0$, we have $2 - x\left|\frac{1}{x}\right| = 2 - x \cdot (-\frac{1}{x}) = 3$. Thus, $\lim_{x\to 0^+} 2 - x\left|\frac{1}{x}\right| = 1 \neq 3 = \lim_{x\to 0^-} 2 - x\left|\frac{1}{x}\right|$. ■

   (d) $\lim_{x\to 0}(x - |x|)h(x)$.

   Solution. Note that for $x > 0$, we have $(x - |x|)h(x) = (x - x)h(x) = 0$. For $x < 0$: $(x - |x|)h(x) = (x - (-x))h(x) = 2x \cdot 0 = 0$. So $(x - |x|)h(x) = 0$ for all x, except for $x = 0$ which is not in the domain of the Heaviside function. But that's not a problem, and so the limit exists and is 0. ■

2. **Basic rules.** Let $c \in \mathbb{R}$ and let f, g be functions such that for some $a \in \mathbb{R}$, both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then:

   1. $\lim_{x\to a} c \cdot f(x) = c \lim_{x\to a} f(x)$
   2. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
   3. $\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$
   4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided that $\lim_{x\to a} g(x) \neq 0$.

   Example. Let's find the following limits.

   (a) $\lim_{x\to 3} \frac{x^2 - x + 6}{\sqrt{x+6}} = \frac{9-3+6}{\sqrt{9}} = \frac{12}{3} = 4$. ■

   (b) $\lim_{x\to\frac{\pi}{6}} \dfrac{11 - \frac{2\pi}{x}}{2\sin(x)} = \dfrac{11 - \frac{2\pi}{\frac{\pi}{6}}}{2 \cdot \frac{1}{2}} = \frac{11-12}{1} = -1$. ■

3. Indeterminate forms. While evaluating $\lim_{x\to a} f(x)$, plugging $x = a$ directly may yield "values" such as

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty - \infty.$$

These are indeterminate, and to get rid of them, we need to be more creative with algebraic tricks.

Example. Find the following limits.

(a) $\lim_{x\to -1} \frac{x^4+6x^2-7}{x+1}$.

Solution. Plugging gives $\frac{0}{0}$. But

$$\lim_{x\to -1}\frac{x^4+6x^2-7}{x+1} = \lim_{x\to -1}\frac{(x^2-1)(x^2+7)}{x+1} = \lim_{x\to -1}\frac{(x-1)\cancel{(x+1)}(x^2+7)}{\cancel{x+1}} = -16. \quad \blacksquare$$

(b) $\lim_{x\to\pi/4}\frac{1-\cot(x)}{\cos(2x)}$.

Solution. Plugging gives $\frac{0}{0}$. But, we can use $\cot(x) = \frac{\cos(x)}{\sin(x)}$ and the double angle identity $\cos(2x) = \cos^2(x) - \sin^2(x) = (\cos(x)-\sin(x))(\cos(x)+\sin(x))$ to get:

$$\lim_{x\to\pi/4}\frac{1-\cot(x)}{\cos(2x)} = \lim_{x\to\pi/4}\frac{\sin(x)-\cos(x)}{\sin(x)\cos(2x)} = \lim_{x\to\pi/4}\frac{\cancel{\sin(x)-\cos(x)}}{\sin(x)\cancel{(\cos(x)-\sin(x))}(\cos(x)+\sin(x))}$$

$$= \frac{-1}{\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)} = -1. \quad \blacksquare$$

(c) $\lim_{x\to 1}\dfrac{\sqrt{1+x}-\sqrt{1+x^2}}{x^3-1}$.

Solution. Plugging gives $\frac{0}{0}$. But, we can multiply top and bottom by $\sqrt{1+x}+\sqrt{1+x^2}$ (which is the *conjugate* of the numerator), and factor the denominator as $(x^3-1) = (x-1)(x^2+x+1)$, to get:

$$\lim_{x\to 1}\frac{\sqrt{1+x}-\sqrt{1+x^2}}{x^3-1} = \lim_{x\to 1}\frac{x(1-x)}{(x-1)(x^2+x+1)(\sqrt{1+x}+\sqrt{1+x^2})}$$

$$= \lim_{x\to 1}\frac{-x\cancel{(x-1)}}{\cancel{(x-1)}(x^2+x+1)(\sqrt{1+x}+\sqrt{1+x^2})} = -\frac{1}{6\sqrt{2}}. \quad \blacksquare$$

(d) $\lim_{x\to 0}\dfrac{\sin^2(x)}{1-\cos(\frac{x}{2})}$.

Solution. Plugging gives $\frac{0}{0}$. But we can multiply top and bottom by $1+\cos(\frac{x}{2})$ (which is the conjugate of the denominator); we get:

$$\lim_{x\to 0}\frac{\sin^2(x)}{1-\cos(\frac{x}{2})} = \lim_{x\to 0}\frac{\sin^2(x)(1+\cos(\frac{x}{2}))}{1-\cos^2(\frac{x}{2})} = \lim_{x\to 0}\frac{4\cancel{\sin^2(\frac{x}{2})}\cos^2(\frac{x}{2})(1+\cos(\frac{x}{2}))}{\cancel{\sin^2(\frac{x}{2})}} = 8. \quad \blacksquare$$

(e) $\lim_{x\to\frac{1}{2}}\dfrac{|1-2x|-2|x-1|+1}{4x^2-1}$.

Solution. For functions involving absolute values, it is often the safest to find the left and right limits separately.

Let $f(x) = \frac{|1-2x|-2|x-1|+1}{4x^2-1}$. For $x < \frac{1}{2}$, we have $|1-2x| = 1-2x$, $|x-1| = 1-x$, and so $f(x) = \frac{(1-2x)-2(1-x)+1}{4x^2-1} = \frac{0}{4x^2-1} = 0$. In other words, $f(x) = 0$ for all $x < \frac{1}{2}$, and so $\lim_{x\to\frac{1}{2}^-} f(x) = 0$.

For $x > \frac{1}{2}$, we have $|1-2x| = 2x-1$, $|x-1| = 1-x$, and so $f(x) = \frac{(2x-1)-2(1-x)+1}{4x^2-1} = \frac{4x-2}{4x^2-1}$. Now, plugging $x = \frac{1}{2}$ gives $\frac{0}{0}$. But

$$\lim_{x\to\frac{1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}^+} \frac{4x-2}{4x^2-1} = \lim_{x\to\frac{1}{2}^+} \frac{2\cancel{(2x-1)}}{\cancel{(2x-1)}(2x+1)} = 1.$$

Hence, $\lim_{x\to\frac{1}{2}^-} f(x) \neq \lim_{x\to\frac{1}{2}^+} f(x)$, and $\lim_{x\to\frac{1}{2}} f(x)$ does not exist. ■

4. **Limit squeeze theorem.** If $g_1(x) \leq f(x) \leq g_2(x)$ near $x = a$, and $\lim_{x\to a} g_1(x) = \lim_{x\to a} g_2(x) = L$, then $\lim_{x\to a} f(x) = L$.

   Example. Find the following limits.

   (a) $\lim_{x\to 0} |x| \sin\left(\frac{1}{x^2}\right)$.

   Solution. Since $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$, we get $-|x| \leq |x|\sin\left(\frac{1}{x^2}\right) \leq |x|$. So $\lim_{x\to 0} |x| \sin\left(\frac{1}{x^2}\right) = 0$ because $\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$. ■

   (b) $\lim_{x\to 2^+}(1-\sin(\frac{\pi}{x}))\cos(\tan(\frac{\pi}{x}))$.

   Solution. Since $-1 \leq \cos(\tan(\frac{\pi}{x})) \leq 1$, we get

$$-(1-\sin(\tfrac{\pi}{x})) \leq (1-\sin(\tfrac{\pi}{x}))\cos(\tan(\tfrac{\pi}{x})) \leq (1-\sin(\tfrac{\pi}{x})).$$

   Also, $\lim_{x\to 2^+} \sin(\frac{\pi}{x}) = 1$, and so $\lim_{x\to 2^+} -(1-\sin(\frac{\pi}{x})) = \lim_{x\to 2^+}(1-\sin(\frac{\pi}{x})) = 0$. Thus, $\lim_{x\to 2^+}(1-\sin(\frac{\pi}{x}))\cos(\tan(\frac{\pi}{x})) = 0$. ■

9 Limits at infinity, infinite limits, and asymptotes

1. **Limits at infinity.** We are interested in $\lim_{x\to\pm\infty} f(x)$, that is, the limit of a function f as x gets larger and larger (or smaller and smaller).

   A common technique (for rational-type functions) is to factor out the terms of largest degree from numerator and denominator.

   Example. Find:

   (a) $\lim_{x\to\infty} \frac{1-x^2-x^4}{3x^4-x}$.

   Solution. Factor x^4:

$$\lim_{x\to\infty} \frac{1-x^2-x^4}{3x^4-x} = \lim_{x\to\infty} \frac{\cancel{x^4}(\underbrace{\tfrac{1}{x^4}-\tfrac{1}{x^2}}_{\to 0}-1)}{\cancel{x^4}(3-\underbrace{\tfrac{1}{x^3}}_{\to 0})} = -\frac{1}{3}. \quad \blacksquare$$

   (b) $\lim_{x\to\infty} \frac{2^x+e^x}{1-e^{x+1}}$.

   Solution. Factor e^x (because $e>2$ and so $e^x > 2^x$):

$$\lim_{x\to\infty} \frac{2^x+e^x}{1-e^{x+1}} = \lim_{x\to+\infty} \frac{\cancel{e^x}\underbrace{\tfrac{1}{(e/2)^x}}_{\to 0}+1}{\cancel{e^x}(\underbrace{\tfrac{1}{e^x}}_{\to 0}-e)} = -\frac{1}{e}. \quad \blacksquare$$

   Exercise. Find $\lim_{x\to\infty} \sqrt{x-1}-\sqrt{x+1}$.

2. **Horizontal Asymptotes.** If L is a *finite* number and $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then $y = L$ is a horizontal asymptote of f.

   Since the two limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ may differ, there may be up to two horizontal asymptotes.

   Example. Find all horizontal asymptotes of $f(x) = \frac{2x-|x|}{\sqrt{1+x^2}}$.

   Solution. We have

$$\lim_{x\to\pm\infty} \frac{2x-|x|}{\sqrt{1+x^2}} = \lim_{x\to\pm\infty} \frac{2x-|x|}{|x|\sqrt{\frac{1}{x^2}+1}}.$$

   So

$$\lim_{x\to+\infty} \frac{2x-|x|}{\sqrt{1+x^2}} = \lim_{x\to+\infty} \frac{\cancel{x}}{\cancel{x}\sqrt{\frac{1}{x^2}+1}} = 1$$

   and

$$\lim_{x\to-\infty} \frac{2x-|x|}{\sqrt{1+x^2}} = \lim_{x\to-\infty} \frac{3\cancel{x}}{-\cancel{x}\sqrt{\frac{1}{x^2}+1}} = -3.$$

   Thus, f has two horizontal asymptotes: $y=1$ (at ∞) and $y=-3$ (at $-\infty$). $\blacksquare$

   Exercise. Find all horizontal asymptotes of $f(x) = \dfrac{\sqrt{\cos\left(\frac{2}{x}\right)-\cos^2\left(\frac{1}{x}\right)}}{\sin\left(\frac{2}{x}\right)}$.

3. **Infinite Limits.** Limits of the form $\dfrac{\pm\infty}{\text{constant}}$ or $\dfrac{\text{constant}}{0}$ are infinite. They evaluate to $\pm\infty$ depending on the sign.

   Example. Find $\lim_{x\to -8^+} \frac{\sqrt[3]{x}+e}{x^2-x-72}$.

   Solution. We have

$$\lim_{x\to -8^+} \frac{\sqrt[3]{x}+e}{x^2-x-72} = \lim_{x\to -8^+} \frac{\overbrace{\sqrt[3]{x}+e}^{\to -2+e>0}}{\underbrace{(x+8)}_{\to 0^+}\underbrace{(x-9)}_{\to -17<0}} = \frac{\text{constant} > 0}{0^-} = -\infty. \quad \blacksquare$$

4. **Vertical Asymptotes.** If a is a *finite* number and $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then $x = a$ is a vertical asymptote of f.

   For rational-type functions, candidates are the roots of the denominator.

   Exercise. Find all vertical asymptotes of $f(x) = \frac{x+3}{x^2-9}$.

   Solution. Candidates: $x = \pm 3$ (roots of the denominator). Now,

$$\lim_{x\to 3^\pm} f(x) = \frac{6}{0^\pm} = \pm\infty$$

   and

$$\lim_{x\to -3^\pm} f(x) = \lim_{x\to -3^\pm} \frac{\cancel{x+3}}{\cancel{(x+3)}(x-3)} = -\frac{1}{6} \neq \pm\infty.$$

   So f has only one vertical asymptote; namely: $x = 3$.

   Note that at $x = -3$, the function is undefined, but the limit exists. This corresponds to a *hole* in the graph of the function. $\blacksquare$

   Exercise. Find the vertical asymptotes of $f(x) = \dfrac{1-\tan(x)}{\cos(2x)}$.

10 Continuity

1. **Definition.** Intuitively, a function is continuous if we can draw its graph without lifting the pen.

   Formally, f is continuous at $x = a$ if:

   (1) $a \in D_f$, meaning $f(a)$ is defined.

   (2) $\lim_{x\to a} f(x)$ exists and equals $f(a)$. More precisely:

   $$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = f(a).$$

   Most familiar functions are continuous on their domains: polynomials, rational functions, square root, absolute value, trigonometric functions (and inverses), exponential, logarithmic, hyperbolic trigonometric, etc.

   Example. $f(x) = \frac{1}{\sqrt{x+1}}$ is continuous everywhere on its domain $(-1, \infty)$. ■

   Example. $f(x) = \tan(x)$ is continuous everywhere except at odd multiples of $\frac{\pi}{2}$, which are not in its domain. ■

   For piecewise-defined functions, even if each expression is continuous in its own piece, we still need to check continuity at the transition points.

   Example. Find all a, b such that

   $$f(x) = \begin{cases} 1 - ax + 2bx^2 & x < -1, \\ b - a + 2 & x = -1, \\ a - 5bx & x > -1 \end{cases}$$

   is continuous.

   Solution. At $x = -1$, we have:

   $$\lim_{x\to -1^-} f(x) = 1 - (-a) + 2b = 1 + a + 2b;$$

   $$\lim_{x\to -1^+} f(x) = a + 5b;$$

   $$f(-1) = b - a + 2.$$

   So:

   $$1 + a + 2b = b - a + 2 \Rightarrow 2a + b = 1;$$

   $$a + 5b = b - a + 2 \Rightarrow 2a + 4b = 2 \Rightarrow a + 2b = 1.$$

   Solve: $a = b = \frac{1}{3}$. ■

   Example. Find all k such that

   $$f(x) = \begin{cases} \sin(\frac{k\pi x}{2\sqrt{3}}) & x < 1, \\ \dfrac{3k^2}{2} & x = 1 \\ \dfrac{|1-x|}{x^2-1} & x > 1 \end{cases}$$

is continuous.

Solution. At $x = 1$, we have

$$\lim_{x\to 1^-} f(x) = \sin(\tfrac{k\pi}{2\sqrt{3}})$$

$$f(1) = \frac{3k^2}{2}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x-1}{(x-1)(x+1)} = \frac{1}{2}.$$

So $\sin(\frac{k\pi}{2\sqrt{3}}) = \frac{3k^2}{2} = \frac{1}{2}$. The second equality implies that $k = \pm\frac{1}{\sqrt{3}}$. But then the first equality implies that $k = \frac{1}{\sqrt{3}}$. ■

2. **Passing limits through continuous functions.** If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to a} f(g(x)) = f\Big(\lim_{x\to a} g(x)\Big) = f(L).$$

We can of course do this with the composition of more than two functions.

Example. Find $\lim_{x\to 0} \cos\left(\dfrac{\pi x}{\sqrt{1+x}-\sqrt{1-x}}\right)$.

Solution. First, we find the limit of the inner function, which is of the form $\frac{0}{0}$, and so we need to multiply by the conjugate:

$$\lim_{x\to 0} \frac{\pi x}{\sqrt{1+x}-\sqrt{1-x}} = \lim_{x\to 0} \frac{\pi x(\sqrt{1+x}+\sqrt{1-x})}{(1+x)-(1-x)} = \lim_{x\to 0} \frac{\pi \cancel{x}(\sqrt{1+x}+\sqrt{1-x})}{2\cancel{x}} = \pi.$$

Now, since $\cos(x)$ is continuous at π, we have

$$\lim_{x\to 0} \cos\left(\frac{\pi x}{\sqrt{1+x}-\sqrt{1-x}}\right) = \cos\left(\lim_{x\to 0} \frac{\pi x}{\sqrt{1+x}-\sqrt{1-x}}\right) = \cos(\pi) = -1. \quad ■$$

Exercise. Find $\lim_{x\to 1} \dfrac{1-2e^{\sin(\pi x)\cos\left(\frac{1}{x}-\pi\right)}}{1+e^{\sin(\pi x)\cos\left(\frac{1}{x}-\pi\right)}}$.

11 Derivatives

1. **Average rate of change.** For a function f, the average rate of change of f between a and b is

$$\frac{f(b)-f(a)}{b-a};$$

which is the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

2. **Derivative at a point.** The derivative of f at a is defined as the limit of the average rate of change between x and a as x approaches a:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Equivalently, if we write $h = x - a$, then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

This is the slope of the tangent line to the graph of f at $(a, f(a))$. It is also the instantaneous rate of change of f at $x = a$.

Example. Let $f(x) = x^3$. Find $f'(1)$.

Solution.

$$f'(1) = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1}(x^2+x+1) = 3. \quad \blacksquare$$

We could instead find a general formula for $f'(x)$ and view it as a new function, the *derivative function* of f:

$$\frac{df}{dx} = \frac{dy}{dx} = f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

For example, when $f(x) = x^3$,

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$$

Example. Let $f(x) = \sqrt{x}$. Find $f'(x)$.

Solution.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}. \quad \blacksquare$$

Example. Let $f(x) = \frac{1}{x}$. Find $f'(x)$.

Solution.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{hx(x+h)} = -\frac{1}{x^2} = -x^{-2}. \quad \blacksquare$$

3. **Power Rule.** We observe a pattern: for every $n \in \mathbb{R}$, $n \neq 0$, if $f(x) = x^n$, then

$$f'(x) = nx^{n-1}.$$

For example, if $f(x) = x^{5/7}$, then

$$f'(x) = \frac{5}{7}x^{-2/7} = \frac{5}{7\sqrt[7]{x^2}}.$$

4. **Differentiability.** Consider the example:

Example. Let $f(x) = |x|$. What is $f'(0)$?

Solution.

$$f'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0}.$$

Now

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1, \qquad \lim_{x \to 0^-} \frac{|x|}{x} = -1.$$

Since the one-sided limits differ, $f'(0)$ does not exist. Geometrically, there is no unique tangent line at $x = 0$. ■

Example. Let $f(x) = \sqrt[3]{x}$. Find $f'(0)$.

Solution. Since $f(x) = x^{1/3}$, the power rule gives

$$f'(x) = \frac{1}{3}x^{-2/3},$$

which is not defined at $x = 0$. Thus $f'(0)$ does not exist (the tangent is vertical). ■

In general, if $f'(a)$ exists, we say f is *differentiable at a*. If $f'(a)$ exists for all $a \in D_f$, then f is *differentiable*.

Every differentiable function is continuous, but **not** every continuous function is differentiable. For example, $f(x) = |x|$ is continuous at 0 but not differentiable.

12 Derivative rules

1. **Basic derivative rules.** We use the following rules quite often:

   1. If $f(x) = c$ for some constant c, then $f'(x) = 0$.
   2. $(cf(x))' = cf'(x)$.
   3. $(f(x) + g(x))' = f'(x) + g'(x)$.
   4. $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.
   5. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.

   All of these can be proved directly from the definition of the derivative. For example:

   Proof of Rule 1. If $f(x) = c$, then

   $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0. \quad \blacksquare$$

   Proof of Rule 4. Let $h(x) = f(x)g(x)$. Then

   $$h'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

   Add and subtract $f(x+h)g(x)$:

   $$h'(x) = \lim_{h\to 0} \frac{f(x+h)(g(x+h) - g(x)) + g(x)(f(x+h) - f(x))}{h}.$$

   Thus

   $$h'(x) = f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x)g'(x) + g(x)f'(x). \quad \blacksquare$$

   Exercise. Prove Rules 2, 3, and 5 directly from the definition of the derivative.

   Example. Find $f'(x)$ when $f(x) = x^7 + x^4 + \frac{1}{\sqrt{x}}$.

   Solution. By the power rule and the sum rule,

   $$f'(x) = 7x^6 + 4x^3 - \frac{1}{2}x^{-3/2}. \quad \blacksquare$$

   Example. Find $f'(x)$ when $f(x) = (x^2 - x^{-2})(x^3 + x^{-3})$.

   Solution. By the power, sum, and product rules,

   $$f'(x) = (2x + 2x^{-3})(x^3 + x^{-3}) + (x^2 - x^{-2})(3x^2 - 3x^{-4}). \quad \blacksquare$$

   Example. Find $f'(x)$ when $f(x) = \dfrac{1-x}{x^{11}+1}$.

   Solution. By the quotient rule,

   $$f'(x) = \frac{(-1)(x^{11}+1) - (1-x)(11x^{10})}{(x^{11}+1)^2}. \quad \blacksquare$$

2. Chain Rule. If $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Example. Find $f'(x)$ when $f(x) = (x^3 + 1)^{-8}$.

Solution.

$$f'(x) = -8(x^3 + 1)^{-9} \cdot (3x^2) = -24x^2(x^3 + 1)^{-9}. \quad \blacksquare$$

Example. Find $f'(x)$ when $f(x) = (1 - 4x)\sqrt[4]{1 + x}$.

Solution.

$$f'(x) = (-4)\sqrt[4]{1 + x} + (1 - 4x) \cdot \frac{1}{4}(1 + x)^{-3/4}. \quad \blacksquare$$

Exercise. Find $f'(x)$ when $f(x) = \sqrt[7]{x + \frac{1}{x}}$.

3. Derivatives of trig functions.

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x, \quad (\tan x)' = \sec^2 x,$$

$$(\csc x)' = -\csc x \cot x, \quad (\sec x)' = \sec x \tan x, \quad (\cot x)' = -\csc^2 x.$$

To prove these, we use the key limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

It follows that

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{-\sin^2 x}{x(\cos x + 1)} = 0.$$

Now the derivative proofs:

<u>*Proof of* $(\sin x)' = \cos x$.</u>

$$f'(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}.$$

Using $\sin(x + h) = \sin x \cos h + \cos x \sin h$:

$$f'(x) = \sin x \cdot \cancelto{0}{\lim_{h \to 0} \frac{\cos h - 1}{h}} + \cos x \cdot \cancelto{1}{\lim_{h \to 0} \frac{\sin h}{h}} = \cos x.$$

<u>*Proof of* $(\cos x)' = -\sin x$.</u>

$$f'(x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h}.$$

Using $\cos(x + h) = \cos x \cos h - \sin x \sin h$:

$$f'(x) = \cos x \cdot \cancelto{0}{\frac{\cos h - 1}{h}} - \sin x \cdot \cancelto{1}{\frac{\sin h}{h}} = -\sin x.$$

Proof of $(\tan x)' = \sec^2 x$.

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x.$$

Example. Find $f'(x)$ when $f(x) = \csc(\sec x)$.

Solution. By the chain rule, we have:

$$f'(x) = -\csc(\sec x)\cot(\sec x) \cdot \sec x \tan x. \quad \blacksquare$$

Example. Find $f'(x)$ when $f(x) = \tan\left(\dfrac{3}{(1-5x)^2}\right)$.

Solution. By the chain rule, we have:

$$f'(x) = \sec^2\left(\frac{3}{(1-5x)^2}\right) \cdot \left(-6(1-5x)^{-3}\right) \cdot (-5). \quad \blacksquare$$

4. Derivatives of exponential functions.

Euler's constant e is the unique base for which

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

Thus,

$$(e^x)' = e^x.$$

For a general base $a > 0$, since $a = e^{\ln a}$,

$$a^x = e^{(\ln a)x}$$

and the chain rule gives

$$(a^x)' = \ln(a)\, a^x.$$

Example. Let $f(x) = 3^{2^{1-\tan x}}$. Find $f'(x)$.

Solution. By the chain rule, we have:

$$f'(x) = \left(\ln 3 \cdot 3^{2^{1-\tan x}}\right) \cdot \left(\ln 2 \cdot 2^{\,1-\tan x}\right) \cdot (-\sec^2 x). \quad \blacksquare$$

Exercise. Find the derivatives of the hyperbolic trigonometric functions. (You do not need to memorize these.)

13 Implicit and logarithmic differentiation

1. **Implicit differentiation.** So far, we have mostly worked with *explicit functions*; equalities of the form $y = f(x)$, with y alone on the left-hand side. But what if y' is needed when the relation between x and y is given, for example, by $x + \sqrt{x}y + y^{12} = 7$, which is an equation with both x, y mixed up?

   This is called an *implicit function*. The idea is: take the derivative of both sides with respect to x, treating y as $f(x)$, and applying the chain rule to any term that involves y. Then solve the resulting equation for y'.

   Example. Find the slope of the tangent line to the circle $x^2 + y^2 = 2$ at $(1, 1)$.

   Solution. Differentiating both sides: $2x + 2y \cdot y' = 0$, and so $y' = -\frac{x}{y}$. At $(1, 1)$, we have $y' = -\frac{1}{1} = -1$. Thus, the tangent line has slope -1. ■

   Example. Find y' if:

   (a) $x + \sqrt{x}y + y^{12} = 7$.

   Solution. Differentiating both sides: $1 + \frac{1}{2\sqrt{x}}y + \sqrt{x}y' + 12y^{11}y' = 0$. So

   $$y'\left(\sqrt{x} + 12y^{11}\right) = -1 - \frac{y}{2\sqrt{x}} \Rightarrow y' = \frac{-1 - \frac{y}{2\sqrt{x}}}{\sqrt{x} + 12y^{11}}. \quad ■$$

   (b) $e^y + x\cos(xy) = 1$.

   Solution. Differentiating both sides: $e^y y' + \cos(xy) - x\sin(xy)(y + xy') = 0$. So

   $$y'(e^y - x^2\sin(xy)) = xy\sin(xy) - \cos(xy) \Rightarrow y' = \frac{xy\sin(xy) - \cos(xy)}{e^y - x^2\sin(xy)}. \quad ■$$

2. **Application I: Derivative of logarithmic functions.** Let $y = \log_a(x)$. Then $a^y = x$, and by implicit differentiation: $a^y\ln(a) \cdot y' = 1 \Rightarrow y' = \frac{1}{a^y\ln(a)} = \frac{1}{x\ln(a)}$. So

   $$(\log_a(x))' = \frac{1}{x\ln(a)}.$$

   In particular, if $a = e$, then $f(x) = \ln(x)$ and

   $$(\ln(x))' = \frac{1}{x}.$$

   More generally, if $g(x) = \ln(f(x))$, then by the chain rule:

   $$g'(x) = \frac{f'(x)}{f(x)}.$$

   Example. Let $f(x) = \dfrac{\ln(\frac{1}{x})}{x}$. Find $f'(x)$.

   Solution. Write $f(x) = \frac{\ln(1/x)}{x} = \frac{-\ln(x)}{x}$. Then

   $$f'(x) = \frac{-(\frac{1}{x}) \cdot x - (-\ln(x)) \cdot 1}{x^2} = \frac{-1 + \ln(x)}{x^2}. \quad ■$$

3. Application II: Derivative of inverse trigonometric functions.

$$(\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}}, \quad (\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}, \quad (\tan^{-1}(x))' = \frac{1}{1+x^2}.$$

<u>Proof of $\sin^{-1}(x)$.</u> Let $y = \sin^{-1}(x)$. Then $\sin(y) = x$. By implicit differentiation, we have:

$$\cos(y)\, y' = 1 \quad \Rightarrow \quad y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}.$$

<u>Proof of $\tan^{-1}(x)$.</u> Let $y = \tan^{-1}(x)$. Then $\tan(y) = x$. By implicit differentiation, we have:

$$\sec^2(y)\, y' = 1 \quad \Rightarrow \quad y' = \cos^2(y) = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}.$$

Exercise. Prove that $(\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}$.

Example. Let $f(x) = \tan^{-1}(e^{-x^2})$. Find $f'(x)$.

Solution. Note that $f(x) = (\tan^{-1}) \circ (e^x) \circ (-x^2)$. So, by the chain rule, we have:

$$f'(x) = \frac{1}{1+(e^{-x^2})^2} \cdot e^{-x^2} \cdot (-2x) = \frac{-2xe^{-x^2}}{1+e^{-2x^2}}. \quad \blacksquare$$

4. Application III: Logarithmic differentiation. To find the derivative of a function of the form $y = f(x)^{g(x)}$, we can take ln of both sides, and then differentiate using implicit differentiation.

Example. Let $f(x) = x^{\frac{1}{x}}$. Find $f'(x)$.

Solution. Set $y = x^{\frac{1}{x}}$. Then $\ln(y) = \ln(x^{\frac{1}{x}}) = \frac{\ln(x)}{x}$. By implicit differentiation:

$$\frac{y'}{y} = \frac{\frac{1}{x}\cdot x - (\ln(x))\cdot 1}{x^2} = \frac{1-\ln(x)}{x^2}. \Rightarrow y' = y \cdot \frac{1-\ln(x)}{x^2} = x^{\frac{1}{x}}\left(\frac{1-\ln(x)}{x^2}\right). \quad \blacksquare$$

Example. Let $f(x) = \cos(x)^{\tan(x)}$. Find $f'(x)$.

Solution. Set $y = \cos(x)^{\tan(x)}$. Then $\ln(y) = \ln(\cos(x)^{\tan(x)}) = \tan(x)\ln(\cos(x))$. By implicit differentiation:

$$\frac{y'}{y} = \sec^2(x)\ln(\cos(x)) + \tan(x)\cdot\frac{-\sin(x)}{\cos(x)} = \sec^2(x)\ln(\cos(x)) - \tan^2(x).$$

So

$$y' = \cos(x)^{\tan(x)}\Big(\sec^2(x)\ln(\cos(x)) - \tan^2(x)\Big). \quad \blacksquare$$

We can also use logarithmic differentiation to handle complicated products and quotients.

Example. Let

$$f(x) = \frac{e^{\sin(x)}(1-x^3)^2}{\sqrt{x\cos(x)}}.$$

Solution. Set $y = f(x)$. Then taking the ln of both sides and using the logarithm rules, we obtain:

$$\ln(y) = \ln(e^{\sin(x)}) + \ln((1-x^3)^2) - \ln(x^{\frac{1}{2}}) - \ln((\cos(x))^{\frac{1}{2}})$$

and so

$$\ln(y) = \sin(x) + 2\ln(1-x^3) - \tfrac{1}{2}\ln(x) - \tfrac{1}{2}\ln(\cos(x)).$$

By implicit differentiation:

$$\frac{y'}{y} = \cos(x) + 2 \cdot \frac{-3x^2}{1-x^3} - \frac{1}{2x} - \frac{1}{2} \cdot \frac{-\sin(x)}{\cos(x)}.$$

Simplify:

$$\frac{y'}{y} = \cos(x) - \frac{6x^2}{1-x^3} - \frac{1}{2x} + \frac{1}{2}\tan(x).$$

So

$$y' = y \cdot \Big(\cos(x) - \frac{6x^2}{1-x^3} - \frac{1}{2x} + \tfrac{1}{2}\tan(x) \Big) = \left(\frac{e^{\sin(x)}(1-x^3)^2}{\sqrt{x\cos(x)}} \right) \Big(\cos(x) - \frac{6x^2}{1-x^3} - \frac{1}{2x} + \tfrac{1}{2}\tan(x) \Big). \quad \blacksquare$$

14 The Intermediate Value Theorem and Newton's method

1. **Intermediate Value Theorem (IVT).** If $f(x)$ is continuous on an interval $[a, b]$, and if $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then f has a root in $[a, b]$, i.e. $f(x) = 0$ for some $x \in [a, b]$.

   More generally: if $f(x)$ is continuous on $[a, b]$, and if $f(a) < k < f(b)$ or $f(b) < k < f(a)$, then $f(x) = k$ has a solution in $[a, b]$.

   Example. Show that $f(x) = x^5 + x - 1$ has a root in $[0, 1]$.

   Solution. At $x = 0$, $f(0) = -1 < 0$. At $x = 1$, $f(1) = 1 + 1 - 1 = 1 > 0$. So by IVT, there exists $c \in [0, 1]$ with $f(c) = 0$. ■

   Example. Show that the equation $\cos(x) = 2x$ has a solution.

   Solution. Let $f(x) = \cos(x) - 2x$. Then $f(0) = 1 > 0$, and $f\left(\frac{\pi}{2}\right) = 0 - \pi < 0$. By IVT, there exists $c \in [0, \frac{\pi}{2}]$ with $f(c) = 0$, i.e. $\cos(c) = 2c$. ■

   Example. Show that the equation $x \sin(x) = 1$ has infinitely many solutions.

   Solution. Let $f(x) = x \sin(x) - 1$.

   • On $[0, \frac{\pi}{2}]$: $f(0) = -1 < 0$, $f(\frac{\pi}{2}) = \frac{\pi}{2} - 1 > 0$, so by IVT, there is one solution in $[0, \frac{\pi}{2}]$.

   • On $[2\pi, 2\pi + \frac{\pi}{2}]$: $f(2\pi) = -1 < 0$, $f(2\pi + \frac{\pi}{2}) = (2\pi + \frac{\pi}{2}) - 1 > 0$, so by IVT, there is one solution in $[2\pi, 2\pi + \frac{\pi}{2}]$.

   • Similarly, on $[4\pi, 4\pi + \frac{\pi}{2}]$, $f(4\pi) = -1 < 0$, $f(4\pi + \frac{\pi}{2}) = (4\pi + \frac{\pi}{2}) - 1 > 0$, so by IVT, there is one solution in $[4\pi, 4\pi + \frac{\pi}{2}]$.

   ... And this goes on and on. For every positive integer n, there is a root in $[2n\pi, 2n\pi + \frac{\pi}{2}]$. Since these intervals are pairwise non-overlapping, it follows that there are infinitely many solutions. ■

   Exercise. Show that the equation $3^x = x^7$ has a solution.

2. **Newton's Method.** Suppose $f(x) = 0$ has a root x^*. Finding this root is not always easy (or even possible). But we can approximate it.

   One way to do that is *Newton's method*, which goes as follows:

   1. Start with an initial guess x_0 near x^*.

   2. The tangent line to $y = f(x)$ at x_0 is $y - f(x_0) = f'(x_0)(x - x_0)$. Its root is

   $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

   3. Repeat the above step on x_1 (instead of x_0) to get x_2, and on x_2 to get x_3, and so on. For $n \geq 0$, we have

   $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

   This sequence $x_0, x_1, x_2, \ldots$ converges to x^*, and further terms are closer and closer to the exact value of x^* (provided x_0 is chosen close enough to x^*).

   Important. If $x_0, x_1, x_2, \ldots$ is **not** converging, then the guess for x_0 was not close enough to the root. So choose a better x_0 that is closer!

Example. Approximate a solution to $x^5 + x - 1 = 0$ correct to six decimals.

Solution. Let $f(x) = x^5 + x - 1$. Then $f'(x) = 5x^4 + 1$. We saw by IVT that f has a root in $[0, 1]$. Take $x_0 = 1$.

$$x_1 = 1 - \frac{1^5 + 1 - 1}{5(1)^4 + 1} = 1 - \frac{1}{6} = 0.833333...$$

$$x_2 = 0.833333 - \frac{0.833333^5 + 0.833333 - 1}{5(0.833333)^4 + 1} = 0.764382...$$

$$x_3 \approx 0.755024..., \quad x_4 \approx 0.754877..., \quad x_5 \approx 0.754877...$$

So $x = 0.754877...$ to six decimals. ■

Example. Approximate the solution to $\cos(x) = 2x$ in $[0, \frac{\pi}{2}]$ correct to seven decimals.

Solution. Let $f(x) = \cos(x) - 2x$. Then $f'(x) = -\sin(x) - 2$. We saw that by IVT, the root exists. Take $x_0 = \frac{1}{2}$.

$$x_1 = 0.5 - \frac{\cos(0.5) - 1}{-\sin(0.5) - 2} \approx 0.450626693...$$

$$x_2 \approx 0.450183674..., \quad x_3 \approx 0.450183611...$$

So $x \approx 0.4501836$ to seven decimals. ■

Example. Approximate $\sqrt{7}$ correct to three decimals using Newton's method.

Solution. We want to solve $f(x) = x^2 - 7 = 0$. Then $f'(x) = 2x$. Take $x_0 = 3$.

$$x_1 = 3 - \frac{3^2 - 7}{2 \cdot 3} = 3 - \frac{2}{6} = \frac{16}{6} = \frac{8}{3} \approx 2.666...$$

$$x_2 = \frac{8}{3} - \frac{\left(\frac{8}{3}\right)^2 - 7}{2 \cdot \frac{8}{3}} = \frac{127}{48} \approx 2.6458333...$$

$$x_3 = \frac{127}{48} - \frac{\left(\frac{127}{48}\right)^2 - 7}{2 \cdot \frac{127}{48}} = \frac{32257}{12192} \approx 2.645751312...$$

Therefore $\sqrt{7} \approx 2.645$ correct to three decimals. ■

15 Mean Value Theorem

1. **Rolle's Theorem.** If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

   That is, the tangent line becomes horizontal somewhere between a and b (a peak or a valley).

   Example. Prove that $e^{2x} = 2\cos(x)$ has exactly one solution in $(0, \frac{\pi}{2})$.

   Solution. Let $f(x) = e^{2x} - 2\cos(x)$. Then $f(0) = 1 - 2 = -1 < 0$ and $f(\frac{\pi}{2}) = e^{\pi} > 0$. Thus, we have $f(0) < 0 < f(\frac{\pi}{2})$, and so by IVT, there exists $x \in (0, \frac{\pi}{2})$ such that $f(x) = 0$.

   Now suppose, for a contradiction, that $f(x) = 0$ has two different solutions a, b in $(0, \frac{\pi}{2})$; say $a < b$. Then $f(a) = f(b) = 0$. Since f is continuous and differentiable on $(a, b) \subset (0, \frac{\pi}{2})$, we can apply Rolle's theorem, deducing that there exists $c \in (a, b) \subset (0, \frac{\pi}{2})$ such that $f'(c) = 0$. It follows that:

   $$f'(c) = 2e^{2c} + 2\sin(c) = 0 \implies e^{2c} + \sin(c) = 0.$$

   But this is impossible because $e^{2c} > 0$ (always), and $\sin(c) > 0$ for $c \in (0, \frac{\pi}{2})$. This contradiction shows that we started with the wrong assumption that there are two different solutions a, b. Hence, there is exactly one solution. ■

2. **Mean Value Theorem (MVT).** If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

   That is, the tangent line at some point between a and b is parallel to the secant line joining $(a, f(a))$ and $(b, f(b))$.

   From this point of view, MVT is just a tilted version of Rolle's theorem!

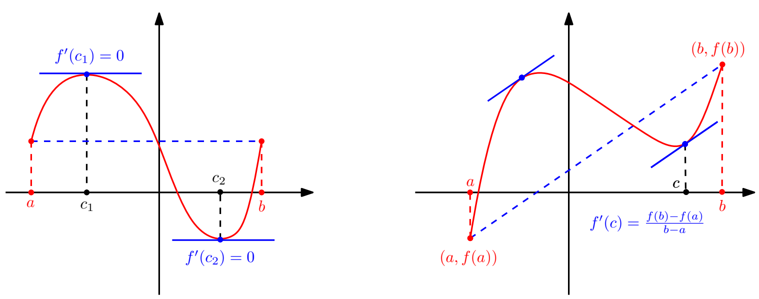


Figure 1: Rolle's Theorem (left) and the Mean Value Theorem (right).

   Proof of MVT. Let

   $$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

   Note that g is continuous on $[a, b]$ and differentiable on (a, b), because f is. Also

   $$g(a) = f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = f(a),$$

   $$g(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = f(a).$$

So $g(a) = g(b)$, and we can apply Rolle's theorem to g to deduce that there exists $c \in (a, b)$ with $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

and so

$$g'(c) = 0 \implies f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \implies f'(c) = \frac{f(b) - f(a)}{b - a}. \quad \blacksquare$$

3. Application of MVT → Increasing and decreasing functions.

We say that a function f is *increasing* on (a, b) if for all $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$.

We say that a function f is *decreasing* on (a, b) if for all $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have $f(x_1) \geq f(x_2)$.

Theorem. *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).*

Proof. Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. By MVT, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $c \in (x_1, x_2) \subset (a, b)$, it follows that $f'(c) \geq 0$. Also, we have $x_2 - x_1 > 0$, and so $f(x_2) - f(x_1) \geq 0$. Thus $f(x_1) \leq f(x_2)$. $\blacksquare$

Exercise. Show that if $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

16 Curve sketching

16.1 Increasing and decreasing functions and critical points

1. **Test for increasing and decreasing functions.** Recall that a function f is said to be *increasing* on an interval (a, b) if for all $x_1, x_2 \in (a, b)$ with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$. Similarly, f is said to be *decreasing* on (a, b) if for all $x_1 \leq x_2$, we have $f(x_1) \geq f(x_2)$.

   Using the Mean Value Theorem, we proved that:

   - If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing on (a, b).
   - If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is decreasing on (a, b).

2. **Critical points.** Given a function f, a point x in the domain of f is called a *critical point* of f if either $f'(x) = 0$ or $f'(x)$ does not exist (that is, f is not differentiable at x).

   Fact. Points at which the sign of the derivative changes (that is, where the increasing/decreasing behavior changes) are always critical points. *But*, not all critical points are points where the sign of the derivative changes (that is, where the increasing/decreasing behavior changes).

   Example. Let $f(x) = x^2$. Determine where f is increasing and where it is decreasing.

   Solution. Since $f'(x) = 2x$, it follows that $f'(x) \leq 0$ for $x \leq 0$, and $f'(x) \geq 0$ for $x \geq 0$. Thus, f is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.

   Note that the change occurs at $x = 0$, which is a critical point because $f'(0) = 0$. ■

   Example. Let $f(x) = |x - 1|$. Determine where f is increasing and where it is decreasing.

   Solution. Since $f(x) = 1 - x$ for all $x < 1$ and $f(x) = x - 1$ for all $x > 1$, it follows that $f'(x) = -1 < 0$ for $x < 1$ and $f'(x) = 1 > 0$ for all $x > 1$. Hence, f is decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$.

   Note that the change occurs at $x = 1$, which is a critical point because f' does not exist at $x = 1$. ■

   Example. Let $f(x) = (x+2)^3 - 1$. Determine where f is increasing and where it is decreasing.

   Solution. Note that $f'(x) = 3(x + 2)^2$, which is never negative. Thus, f is increasing everywhere and never decreasing.

   However, the point $x = -2$ is a critical point since $f'(-2) = 0$ there, but no change in behavior occurs there. ■

3. **The algorithm**. To determine the critical points and where f is increasing or decreasing, follow these steps:

   1. Find all x **in the domain of** f such that $f'(x) = 0$ or $f'(x)$ does not exist. These are the critical points.

   2. Determine the sign of $f'(x)$ on the intervals between consecutive critical points (e.g., using a sign chart). If $f'(x) > 0$, then f is increasing on that interval. If $f'(x) < 0$, then f is decreasing on that interval.

   3. Include the endpoints of the domain in those intervals **only if** f is continuous at those points. An endpoint may belong to both intervals it borders.

Example Find the critical points and determine where f is increasing or decreasing for:

(a) $f(x) = x^6 - 6x^4 + 7$.

Solution. Note that $f'(x) = 6x^5 - 24x^3 = 2x^3(3x^2 - 12)$. So $f'(x) = 0$ for $x = 0$ or $x = \pm 2$. These are the critical points.

Note that the function is continuous at all critical points. (It is continuous everywhere!)

We now figure the sign of $f'(x)$:

x	$-\infty$		-2		0		2		$+\infty$
$f'(x)$		$-$		$+$		$-$		$+$	

Hence, f is

- decreasing on $(-\infty, -2]$ and $[0, 2]$;
- increasing on $[-2, 0]$ and $[2, \infty)$.

Note that while f is decreasing on $(-\infty, -2]$ and $[0, 2]$, it is *not* decreasing on $(-\infty, -2] \cup [0, 2]$. Here is why: Note that $-\sqrt{5}, 1 \in (-\infty, -2] \cup [0, 2]$, $f(-\sqrt{5}) = 125 - 150 + 7 = -18$ and $f(1) = 1 - 6 + 7 = 2$. Thus, we have $-\sqrt{5} < 1$ and $f(-\sqrt{5}) < f(1)$, which is not what a decreasing function should be doing.

So, **watch out!** Don't take the union of the interval over which f has the same behavior. Just write them out separately. ■

(b) $f(x) = \frac{1+x-x^2}{x^2-1}$.

Solution. Note that $D_f = \mathbb{R} \setminus \{-1, 1\}$. Differentiate:

$$f'(x) = \frac{(1-2x)(x^2-1) - (1+x-x^2)(2x)}{(x^2-1)^2} = \frac{-(x^2+1)}{(x^2-1)^2}.$$

Note that f' has no roots, and f' is defined wherever f is defined. So f has no critical points. (Yes, f' is not defined at $x = \pm 1$, but these points are not in D_f).

Also, $x^2 + 1 > 0$ and $(x^2 - 1)^2 > 0$. Therefore, $f'(x) < 0$ for all $x \in D_f$, and so f is decreasing everywhere on its domain. ■

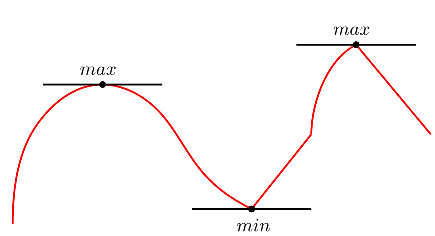
16.2 Local minima and maxima

Let f be a function and $x_0 \in D_f$. We say that x_0 is a *local minimum* of f if there exists an open interval $I \subseteq D_f$ containing x_0 such that

$$f(x) \geq f(x_0) \text{ for all } x \in I.$$

Similarly, x_0 is a *local maximum* of f if there exists an open interval $I \subseteq D_f$ containing x_0 such that

$$f(x) \leq f(x_0) \text{ for all } x \in I.$$



Fact. Local minima and maxima occur only at *critical points*, but not all critical points are local minima or maxima. For instance, $x = 1$ is a critical point for $f(x) = (x-1)^3$, but it is neither a local minimum nor a local maximum.

So how do we tell if a critical point of f is indeed a local minimum or maximum?

First derivative test. Let c be a critical point of a function f at which f is continuous.

- If the sign of $f'(x)$ changes *from positive to negative* as x passes through c from left to right, then f has a local **maximum** at $x = c$.
- If the sign of $f'(x)$ changes *from negative to positive* as x passes through c, then f has a local **minimum** at $x = c$.
- If the sign of $f'(x)$ *does not change* as x passes through c, then f has **neither** a local minimum nor a local maximum at $x = c$.

Example Find all local minima and maxima of

(a) $f(x) = 11 + 5x^3 - 5x^4 + x^5$.

Solution. We have $f'(x) = 15x^2 - 20x^3 + 5x^4 = 5x^2(x^2 - 4x + 3) = 5x^2(x-1)(x-3)$. So the critical points are $x = 0, 1, 3$, and f is continuous at all of them.

Now, we figure out the sign of $f'(x)$:

x	$-\infty$		0		1		3		$+\infty$
$f'(x)$		$+$		$+$		$-$		$+$	

Hence, f has a local maximum at $x = 1$ and a local minimum at $x = 3$. Note even though $x = 0$ is a critical point, f has neither a maximum nor a minimum there. ∎

(b) $f(x) = \frac{2-x}{(x+1)^2}$.

Solution. We have $D_f = \mathbb{R} \setminus \{-1\}$. Also,

$$f'(x) = \frac{-(x+1)^2 - (2-x)(2)(x+1)}{(x+1)^4} = \frac{-(x+1)((x+1) + 2(2-x))}{(x+1)^4} = \frac{-(x+1)(5-x)}{(x+1)^4} = \frac{x-5}{(x+1)^3}.$$

So the only critical point is $x = 5$ (where $f'(5) = 0$). Note that $x = -1$ is *not* critical because it doesn't belong to the domain of f.

Now we check the sign of $f'(x)$:

x	$-\infty$		-1		5		$+\infty$
$f'(x)$		$+$		$-$		$+$	

Thus, f has a local minimum at $x = 5$. Note that $x = -1$ is neither a maximum nor a minimum, because it is not even a critical point. ∎

(c) $f(x) = \dfrac{1}{1 - e^{1-x^2}}$.

Solution. Note that $1 - e^{1-x^2} = 0 \implies x = \pm 1$. So $D_f = \mathbb{R} \setminus \{\pm 1\}$. Also, we have

$$f'(x) = \frac{2xe^{1-x^2}}{(1 - e^{1-x^2})^2}.$$

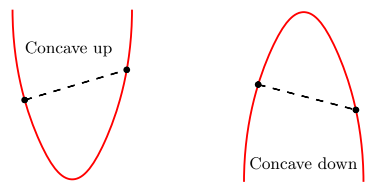
So the only critical point is $x = 0$. Note that the roots $x = \pm 1$ of the denominator of f' are not critical points as they don't belong to the domain of f.

Since e^{1-x^2} and $(1 - e^{1-x^2})^2$ are positive everywhere, the sign of $f'(x)$ is the same as the sign of x. Hence, $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$, and so $x = 0$ is a local minimum. ∎

16.3 Concavity

1. **Definitions and the test for concavity.** A function f is said to be *concave up* on an interval I if for all $x_1, x_2 \in I$ with $x_1 < x_2$, the secant line through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies entirely *above* the graph of f between x_1 and x_2.

   Analogously, f is said to be *concave down* on I if for all $x_1 < x_2$ in I, the secant line through $(x_1, f(x_1))$ and $(x_2, f(x_2))$ lies entirely *below* the graph of f between x_1 and x_2.

   Observation. When the slope of the tangent line to the graph of f, that is, the function $f'(x)$, is increasing, f is concave up; and when $f'(x)$ is decreasing, f is concave down.

   To determine whether $f'(x)$ is increasing or decreasing, we look at its derivative, that is, the *derivative of the derivative of* f, called the *second derivative of* f, denoted by $f''(x)$, or $\frac{d^2f}{dx^2}$, or $\frac{d^2y}{dx^2}$.

   Test for Concavity.

   - If $f''(x) \geq 0$ for all $x \in (a, b)$, then f is concave up on (a, b).
   - If $f''(x) \leq 0$ for all $x \in (a, b)$, then f is concave down on (a, b).

   In particular, intervals of concavity are separated by points x at which $f''(x) = 0$ or $f''(x)$ does not exist.

2. **Points of inflection.** A point $x = c$ in the domain of f is called a *point of inflection* of f if the concavity of f changes as x passes through $x = c$.

   For finding points of inflection and intervals of concavity, follow this algorithm:

   1. Find all points x **in the domain of** f such that $f''(x) = 0$ or $f''(x)$ does not exist.

   2. Determine the sign of $f''(x)$ around those points. If $f''(x) > 0$, then f is concave up; if $f''(x) < 0$, then f is concave down. Include endpoints only if f is continuous there. As before, do **not** merge intervals that have the same concavity; list them separately.

   3. The points obtained in Step 1 around which the sign of $f''(x)$ **changes** are the inflection points of f.

   Example. Find the intervals of concavity and points of inflection of

   (a) $f(x) = -x^4 - e^{-x}$.

   Solution. We have $D_f = \mathbb{R}$, and

   $$f'(x) = -4x^3 + e^{-x}, \quad f''(x) = -12x^2 - e^{-x}.$$

   Since $f''(x) = -12x^2 - e^{-x} < 0$ for all x, it never changes sign. Therefore, f is concave down on $\mathbb{R}$ and has no points of inflection. ■

(b) $f(x) = \frac{2}{3}x^6 - \frac{5}{2}x^4 + 4x - 1$.

Solution. We have $D_f = \mathbb{R}$, and

$$f'(x) = 4x^5 - 10x^3 + 4, \qquad f''(x) = 20x^4 - 30x^2 = 10x^2(2x^2 - 3).$$

Thus, $f''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$, and f is continuous at all of them.

Now we analyze the sign of $f''(x)$:

x	$-\infty$		$-\sqrt{\frac{3}{2}}$		0		$\sqrt{\frac{3}{2}}$		$+\infty$
$f''(x)$		$+$		$-$		$-$		$+$	

Hence, f is concave down on $(-\infty, -\sqrt{\frac{3}{2}}]$ and $[\sqrt{\frac{3}{2}}, \infty)$ and concave up on $[-\sqrt{\frac{3}{2}}, 0]$ and $[0, \sqrt{\frac{3}{2}}]$.

Therefore, $x = -\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}$ are points of inflection. Note that $x = 0$ is *not* a point of inflection, because the sign of f'' doesn't change around it. ■

3. The Second derivative test. Concavity also gives us a new way to spot local minima and maxima at the roots of the derivative of a function f.

Let f be a function and $c \in D_f$ such that $f'(c) = 0$. Then:

- If $f''(c) > 0$, then f has a local minimum at $x = c$.
- If $f''(c) < 0$, then f has a local maximum at $x = c$.
- If $f''(c) = 0$, the test is inconclusive (it could be a minimum, a maximum, or neither).

Example. Let $f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4 - x^3 + \frac{1}{2}x^2 + 2x$. Determine which critical points are local minima, local maxima, or neither.

Solution. We have

$$f'(x) = x^4 - x^3 - 3x^2 + x + 2 = (x+1)^2(x-1)(x-2).$$

So the roots of f' are $x = -1, 1, 2$. Also,

$$f''(x) = 4x^3 - 3x^2 - 6x + 1.$$

Evaluate at the critical points:

$$f''(2) = 32 - 12 - 12 + 1 = 9 > 0 \;\Rightarrow\; \text{Local minimum at } x = 2,$$

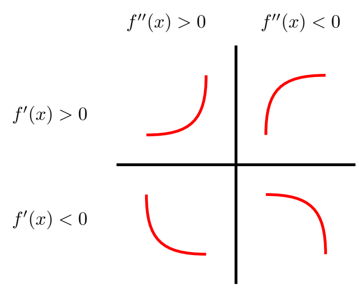
$$f''(1) = 4 - 3 - 6 + 1 = -4 < 0 \;\Rightarrow\; \text{Local maximum at } x = 1,$$

$$f''(-1) = -4 - 3 + 6 + 1 = 0 \;\Rightarrow\; \text{Inconclusive at } x = -1. \quad ■$$

16.4 Curve sketching

Now we put together these ingredients and devise a recipe for sketching the graphs of functions:

1. Find the domain of f.
2. Find the vertical asymptotes: all a such that $\lim_{x \to a^{\pm}} f(x) = \pm\infty$.
3. Find the horizontal asymptotes: finite numbers L such that $\lim_{x \to \pm\infty} f(x) = L$.
4. Find all x for which $f'(x) = 0$ or $f'(x)$ does not exist (possible critical points).
5. Find all x for which $f''(x) = 0$ or $f''(x)$ does not exist (possible inflection points).
6. Decide the increasing/decreasing behavior and concavity on each interval determined by the points from (4) and (5). List the local extrema (by first or second derivative test) and points of inflection (where concavity truly changes).
7. For each interval, sketch using the sign chart of f' and f'':



Example. Sketch the graph of $f(x) = x^4 - 2x^2 + 7$.

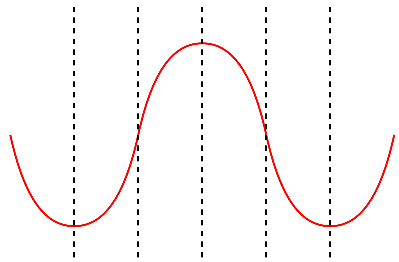
Solution.

1. Domain: $D_f = \mathbb{R}$.
2. Vertical asymptotes: None.
3. Horizontal asymptotes: None.
4. $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x-1)(x+1)$. Critical points: $x = -1, 0, 1$.
5. $f''(x) = 12x^2 - 4 = 4(3x^2 - 1)$. So $f''(x) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{3}}$.
6. Signs of f' and f'':

x	$-\infty$		-1		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		1		$+\infty$
f'		$-$		$+$		$+$		$-$		$-$		$+$	
f''		$+$		$+$		$-$		$-$		$+$		$+$	

In particular, $x = \pm\frac{1}{\sqrt{3}}$ are both points of inflection.

7. The graph looks like this:



■

Example. Sketch the graph of $f(x) = \dfrac{x^2}{9-x^2}$.

Solution.

1. Domain: $\mathbb{R} \setminus \{-3, 3\}$.

2. Vertical asymptotes:

$$\lim_{x\to 3^-} f(x) = +\infty, \quad \lim_{x\to 3^+} f(x) = -\infty,$$
$$\lim_{x\to -3^-} f(x) = -\infty, \quad \lim_{x\to -3^+} f(x) = +\infty.$$

So $x = \pm 3$.

3. Horizontal asymptotes. $\lim_{x\to\pm\infty} \frac{x^2}{9-x^2} = -1$. So $y = -1$.

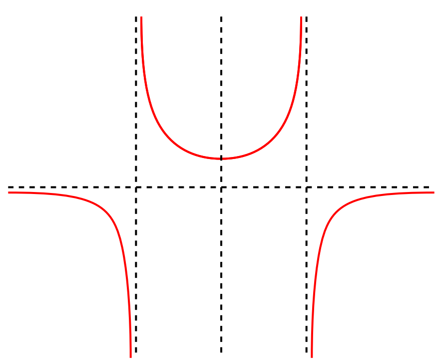
4. $f'(x) = \frac{18x}{(9-x^2)^2}$. The only critical point is $x = 0$ (At $x = \pm 3$, $f'(x)$ doesn't exist.)

5. $f''(x) = \frac{54(3+x^2)}{(9-x^2)^3}$. No roots since $54(3+x^2) > 0$. But $f''(x)$ doesn't exist at $x = \pm 3$.

6. Signs of f' and f'':

x	$-\infty$		-3		0		3		$+\infty$
f'		$-$		$-$		$+$		$+$	
f''		$-$		$+$		$+$		$-$	

7. The graph looks like this:



■

Example. Sketch the graph of $f(x) = e^{1-\frac{1}{x}}$ using the full recipe.

Solution.

1. Domain: $\mathbb{R} \setminus \{0\}$.

2. Vertical asymptotes: $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 0^-} f(x) = +\infty$. So $x = 0$ only from the left.

3. Horizontal asymptotes: $\lim_{x \to \pm\infty} e^{1-\frac{1}{x}} = e$. So $y = e$.

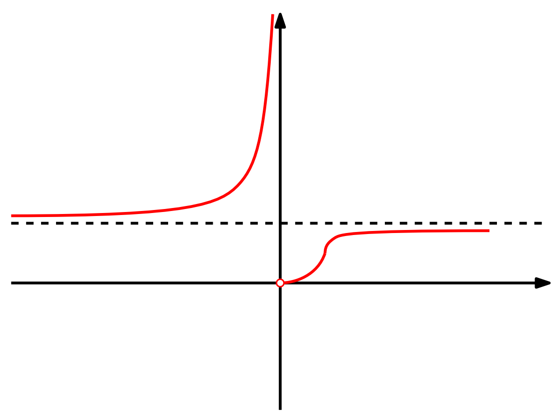
4. $f'(x) = \dfrac{1}{x^2} \cdot e^{1-\frac{1}{x}} > 0$. No critical points. (At $x = 0$, $f'(x)$ doesn't exist.)

5. $f''(x) = \dfrac{1-2x}{x^4} \cdot e^{1-\frac{1}{x}}$. So $f''(x) = 0 \implies x = \frac{1}{2}$. (At $x = 0$, $f''(x)$ doesn't exist).

6. Signs of f' and f'':

x	$-\infty$		0		$\frac{1}{2}$		$+\infty$
f'		$+$		$+$		$+$	
f''		$+$		$+$		$-$	

7. The graph looks like this:



■

17 Global minima and maxima, and optimization

A function f has a *global maximum* on an interval I at $x_0 \in I$ if for every $x \in I$,

$$f(x) \leq f(x_0).$$

Similarly, f has a *global minimum* on I at x_0 if for every $x \in I$,

$$f(x) \geq f(x_0).$$

Fact. If f is continuous over a closed interval $[a, b]$, then f attains both a global maximum and a global minimum on $[a, b]$. These occur either at a critical point of f in (a, b), or at one of the endpoints a and b.

So, to find the global extrema of a continuous function f on $[a, b]$:

1. Find all critical points of f in $[a, b]$ (i.e., all $c \in [a, b]$ where $f'(c) = 0$ or $f'(c)$ does not exist).
2. Compute $f(a)$, $f(b)$, and $f(c)$ for all critical points $c \in [a, b]$ of f.
3. The largest value is the global maximum; the smallest is the global minimum.

Example. Find the global extrema of $f(x) = 3x^5 - 50x^3 + 135x - 71$ over $[0, 4]$.

Solution. We have $f'(x) = 15x^4 - 150x^2 + 135 = 15(x^4 - 10x^2 + 9) = 15(x^2 - 1)(x^2 - 9)$. So $x = 1$ and $x = 3$ are the critical points in $[0, 4]$.
Now, $f(0) = -71$, $f(1) = 17$, $f(3) = -287$ and $f(4) = 341$. Thus the global minimum is -287 at $x = 3$ and the global maximum is 341 at $x = 4$. ■

Example. You have 6m of fencing to enclose a rectangular patio along a wall, so only three sides need fencing. What is the maximum area you can enclose?

Solution. **Follow this procedure:**

1. If needed, draw a picture and identify the variables.
2. Identify the quantity to be maximized/minimized, and the constraints.
3. Rewrite that quantity in terms of one of the variables using the constraints.
4. Find the global extrema as required.

Here,

1. Let the perpendicular sides of the patio have length a and let the open side have length b.

2. The area enclosed is $S = ab$, and the constraint is $2a + b = 6$; in particular, we have $a \in [0, 3]$.

3. Rewrite the area as a function of a only: $f(a) = S = ab = a(6 - 2a)$.

4. Find the global maximum of $f(a)$ for $a \in [0, 3]$. We have $f'(a) = 6 - 4a$. So, there is a critical point at $a = \frac{3}{2}$. Now, $f(0) = 0$, $f\left(\frac{3}{2}\right) = 6\left(\frac{3}{2}\right) - 2\left(\frac{3}{2}\right)^2 = 9 - \frac{9}{2} = \frac{9}{2}$ and $f(3) = 0$. Hence, the maximum area is $4.5\,\text{m}^2$, attained when $a = \frac{3}{2}$ and $b = 3$. ■

Example. You have a $10\,\text{m}^2$ tin sheet and want to make a cylindrical can (with top and bottom). What is the maximum volume of the can you can make?

Solution.

1. Let the radius of the top and the bottom be r and let the height be h.

2. The volume is $V = \pi r^2 h$. The constraint comes from the surface area:

$$2\pi r^2 + 2\pi r h = 10 \Rightarrow h = \frac{10 - 2\pi r^2}{2\pi r} = \frac{5}{\pi r} - r.$$

   In particular, note that $r \geq 0$, and if all the sheet goes for the top (bottom) surface – as if the $h = 0$ – then:

$$\pi r^2 = 10 \Rightarrow r = \sqrt{\frac{10}{\pi}}.$$

   So we have $r \in \left[0, \sqrt{\frac{10}{\pi}}\right]$.

3. Rewrite the volume as a function of r only: $f(r) = V = \pi r^2 h = \pi r^2 \left(\frac{5}{\pi r} - r\right) = 5r - r^3\pi$.

4. Find the maximum of $f(r)$ for $r \in [0, \sqrt{10/\pi}]$. We have $f'(r) = 5 - 3\pi r^2 = 0$. So there is a critical point at $r = \sqrt{\frac{5}{3\pi}}$. Now,

$$f(0) = f\left(\sqrt{\tfrac{10}{\pi}}\right) = 0, \quad f\left(\sqrt{\frac{5}{3\pi}}\right) = \frac{10\sqrt{5}}{3\sqrt{3\pi}}.$$

   The maximum volume is $\frac{10\sqrt{5}}{3\sqrt{3\pi}}$ attained when $r = \sqrt{\frac{5}{3\pi}}$ and $h = \frac{5}{\pi r} - r = 2\sqrt{\frac{5}{3\pi}} = 2r$. ∎

Example. Find the point (x, y) on the curve $y = \sqrt{x}$ for $x \in [0, 4]$ that is the closest to $(2, 0)$.

Solution.

1. Variables are just x, y.

2. The distance is $d = \sqrt{(x-2)^2 + (y-0)^2} = \sqrt{x^2 - 4x + 4 + y^2}$.
   The constraints are $y = \sqrt{x}$ and $x \in [0, 4]$.

3. Instead of d, we minimize d^2. And we can rewrite d^2 as a function of x only:

$$f(x) = d^2 = x^2 - 4x + 4 + (\sqrt{x})^2 = x^2 - 3x + 4.$$

4. Find the global minimum of $f(x)$ on $[0, 4]$. We have $f'(x) = 2x - 3$. So there is a critical point at $x = \frac{3}{2}$. Now, $f(0) = 4, f(4) = 8$ and $f(3/2) = 7/4$. Hence, the closest point is $\left(\frac{3}{2}, \sqrt{\frac{3}{2}}\right)$, with minimal distance $\sqrt{\frac{7}{4}}$. ∎

Example. Prove that it is impossible to draw a rectangle inside a triangle (with one side of the rectangle along the base of the triangle) such that the rectangle occupies more than half of the area.

Solution.

1. Let the triangle have base length b and height h. Think of b, h as **fixed**. Let the rectangle have side lengths x and y, inscribed with the x-side on the triangle's base. Think of x, y as the **variables**.

2. The area of the rectangle is $A = xy$. The constraint comes from similarity: By similarity of triangles: $\dfrac{y}{h} = \dfrac{b-x}{b} \Rightarrow y = h(1 - \frac{x}{b})$. Moreover, we have $x \in [0, b]$.

3. Rewrite the area as a function of x only:

$$f(x) = A = xy = hx(1 - \tfrac{x}{b}) = h(x - \tfrac{x^2}{b}).$$

4. Find the maximum of $f(x)$ on $[0, b]$. We have $f'(x) = h(1 - \frac{2x}{b})$. So there is a critical point at $x = \frac{b}{2}$. Now, $f(0) = f(b) = 0$, and $f(\frac{b}{2}) = \frac{bh}{4}$. Thus, the maximum area of the rectangle is $\frac{bh}{4}$, and

$$\frac{\text{Maximum area of the rectangle}}{\text{Area of triangle}} = \frac{\frac{bh}{4}}{\frac{bh}{2}} = \frac{1}{2}. \quad \blacksquare$$

Exercise. If you fit the largest possible cone inside a sphere, what fraction of the sphere's volume is occupied by the cone? (Here the cone is a right circular cone with its base perpendicular to the axis of symmetry.)

18 Related Rates

Suppose two quantities x and y vary with time t. If x and y are related, and we know $\frac{dx}{dt}$ — that is, how quickly x changes with time — then we can often find $\frac{dy}{dt}$, the rate at which y changes with time. We take the following steps:

1. Define your two quantities (draw a picture if needed).
2. Find a formula relating the two quantities.
3. Differentiate implicitly with respect to t.
4. Solve for the desired rate using the known rate(s) and values.

Example. A person is walking away from a street light at a rate of 2 m/s. The person is 2 m tall and the street light is 6 m tall. How quickly is the length of the person's shadow changing?

Solution. Let x be the distance from the person to the light and let s be the length of the shadow. By similar triangles,

$$\frac{6}{x+s} = \frac{2}{s} \quad \Rightarrow \quad 6s = 2x + 2s \Rightarrow 4s = 2x \Rightarrow s = \frac{x}{2}.$$

Differentiate both sides with respect to t:

$$\frac{ds}{dt} = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}(2) = 1.$$

So the shadow is lengthening at a rate of 1 m/s. ■

Example. A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/s?

Solution. Let h be the height of the water and let V be the volume. If water is being drained at a rate of 25 cm^3/s, this means that

$$\frac{dV}{dt} = -25.$$

Also,

$$V = \pi(20)^2 h = 400\pi h,$$

Differentiating both sides with respect to t:

$$\frac{dV}{dt} = 400\pi\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-25}{400\pi} = -\frac{1}{16\pi}.$$

Thus, the water level in the tank is decreasing at a rate of about 0.02 cm/s. ■

Example. A steel cylinder is being compressed in a hydraulic press. The steel remains cylindrical with a constant volume of 40 cm^3. If the height is decreasing at a rate of 1 cm/s, find the rate at which the radius is increasing when $r = 2$ cm.

Solution. Let h be the height and let r be the radius. Then

$$\pi r^2 h = 40.$$

Differentiate both sides with respect to t:

$$\pi(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}) = 0 \Rightarrow \frac{dr}{dt} = -\frac{r^2}{2rh}\frac{dh}{dt} = -\frac{r}{2h}\frac{dh}{dt}.$$

At $r = 2$, we find $h = \frac{40}{\pi r^2} = \frac{40}{4\pi} = \frac{10}{\pi}$, and $\frac{dh}{dt} = -1$.

$$\frac{dr}{dt} = -\frac{2}{2(10/\pi)}(-1) = \frac{\pi}{10}.$$

So the radius is increasing at $\dfrac{\pi}{10}$ cm/s. ■

Example. A ladder 13 m long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at 0.6 m/s. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?

Solution. Let x be the distance from the wall and let y be the height on the wall. Then $x^2+y^2 = 13^2$. Differentiate with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

At $x = 5$, we have $y = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = 12$. Thus,

$$\frac{dy}{dt} = -\frac{5}{12}(0.6) = -\frac{1}{4}.$$

So the top is sliding down at 0.25 m/s. ■

Example. Sand is poured at a constant volume rate of 15 cm^3/s, forming a conical pile whose height always equals its base diameter. How fast is the height increasing when the pile is 3 cm high?

Solution. Let h be the height of the pile, let r be its radius, and let V be its volume. Then we have

$$\frac{dV}{dt} = 15;$$

and since $h = 2r$, we have

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}.$$

Differentiate both sides with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{60}{\pi h^2}.$$

Thus, at $h = 3$, we have

$$\frac{dh}{dt} = \frac{20}{3\pi};$$

meaning that the height of the pile is increasing at $\dfrac{20}{3\pi}$ cm/s. ■

19 L'Hôpital's rule

Suppose f, g are differentiable near $x = a$ (except possibly at a). Assume that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has the indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ and that the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (finite) or is $\pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Notes. The rule can also be applied to:

1. One–sided limits $x \to a^{\pm}$ and for limits at infinity $x \to \pm\infty$.
2. The indeterminate form $0 \cdot \infty$. Rewrite as a quotient: $0 \cdot \infty = \dfrac{0}{1/\infty} = \dfrac{0}{0}$.
3. The indeterminate forms 0^0, $(+\infty)^0$, 1^∞. Take ln, use L'Hôpital, and then exponentiate again.
   - $L = 0^0 \Rightarrow \ln(L) = 0 \cdot \ln(0) = 0 \cdot (-\infty)$.
   - $L = (+\infty)^0 \Rightarrow \ln(L) = 0 \cdot \ln(+\infty) = 0 \cdot \infty$.
   - $L = 1^\infty \Rightarrow \ln(L) = \infty \cdot \ln(1) = \infty \cdot 0$.

Example. Find the following limits.

(a) $\displaystyle\lim_{x\to\pi} \frac{x^2 - \pi^2}{\sin x}$.

Solution. Type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to\pi} \frac{x^2 - \pi^2}{\sin x} = \lim_{x\to\pi} \frac{2x}{\cos x} = \frac{2\pi}{\cos \pi} = \frac{2\pi}{-1} = -2\pi. \quad \blacksquare$$

(b) $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$.

Solution. Type $\frac{\infty}{\infty}$. Apply L'Hôpital's:

$$\lim_{x\to\infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x\to\infty} \frac{4x - 3}{2x + 47} = \lim_{x\to\infty} \frac{4}{2} = 2. \quad \blacksquare$$

(c) $\displaystyle\lim_{x\to 0} \frac{\sec x - 1}{\sin x}$.

Solution. Type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to 0} \frac{\sec x - 1}{\sin x} = \lim_{x\to 0} \frac{\sec x \tan x}{\cos x} = \lim_{x\to 0} \frac{\tan x}{\cos^2 x} = 0. \quad \blacksquare$$

(d) $\displaystyle\lim_{x\to 0^+} \frac{1}{x^2 \ln x}$.

Solution. Write $\frac{1}{x^2 \ln x} = \frac{\frac{1}{x^2}}{\ln x}$. So the type is $\frac{\infty}{\infty}$. Apply L'Hôpital's:

$$\lim_{x\to 0^+} \frac{\frac{1}{x^2}}{\ln x} = \lim_{x\to 0^+} \frac{-\frac{2}{x^3}}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{-2}{x^2} = -\infty. \quad \blacksquare$$

(e) $\lim\limits_{x\to 1^+} x^{\frac{1}{x-1}}$.

Solution. Let L be the limit. Then:

$$\ln L = \ln\left(\lim_{x\to 1^+} x^{\frac{1}{x-1}}\right) = \lim_{x\to 1^+} \ln\left(x^{\frac{1}{x-1}}\right) = \lim_{x\to 1^+} \frac{\ln x}{x-1}.$$

The type is $\frac{0}{0}$. So by L'Hôpital's:

$$\ln L = \lim_{x\to 1^+} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

Hence $L = e^1 = e$. ■

(f) $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$.

Solution. Let L be the limit. Then by L'Hôpital's:

$$\ln L = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} \ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

The type is $\frac{0}{0}$. So by L'Hôpital's:

$$\ln L = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{\cancel{\frac{-1}{x^2}}}{1+\frac{1}{x}}}{\cancel{\frac{-1}{x^2}}} = 1,$$

and thus $L = e$. ■

(g) $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}$

Solution. Type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1} = \lim_{x\to 0} \frac{\frac{2x}{2\sqrt{x^2+1}}}{\frac{1}{2\sqrt{x+1}}} = \lim_{x\to 0} \frac{2x\sqrt{x+1}}{\sqrt{x^2+1}} = 0. \quad ■$$

(h) $\lim\limits_{x\to 1^-} \dfrac{\sqrt[10]{1-x}-1}{\frac{\pi}{2}-\sin^{-1}(x)}$.

Solution. Type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to 1^-} \frac{\sqrt[10]{1-x}-1}{\frac{\pi}{2}-\sin^{-1}(x)} = \lim_{x\to 1^-} \frac{-\frac{1}{10}(1-x)^{-9/10}}{-\frac{1}{\sqrt{1-x^2}}} = \lim_{x\to 1^-} \frac{\sqrt{1-x^2}}{10(1-x)^{9/10}}$$

$$= \lim_{x\to 1^-} \frac{(1-x)^{1/2}(1+x)^{1/2}}{10(1-x)^{9/10}} = \lim_{x\to 1^-} \frac{\cancelto{\sqrt{2}}{(1+x)^{1/2}}}{10\cancelto{0}{(1-x)^{2/5}}} = \infty. \quad ■$$

(i) $\lim_{x\to 0} \dfrac{\tan\bigl(1-e^x\bigr)}{e^x-\cos(x)}$

Solution. Type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to 0} \frac{\tan\bigl(1-e^x\bigr)}{e^x-\cos(x)} = \lim_{x\to 0} \frac{-e^x\sec^2(1-e^x)}{e^x+\sin(x)} = \frac{-\sec^2(0)}{1+0} = -1. \quad \blacksquare$$

(j) $\lim_{x\to 0} \dfrac{\sin(x)-\tan(x)}{1-\sqrt{1-x^3}}$

Solution. Type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to 0} \frac{\cos(x)-\sec^2(x)}{\dfrac{3x^2}{2\sqrt{1-x^3}}} = \lim_{x\to 0} \frac{2\sqrt{1-x^3}\,\bigl(\cos(x)-\sec^2(x)\bigr)}{3x^2}.$$

Note that

$$\lim_{x\to 0} 2\sqrt{1-x^3} = 2.$$

So we just need to find $\lim_{x\to 0} \dfrac{\cos(x)-\sec^2(x)}{3x^2}$, which is again of type $\frac{0}{0}$. Apply L'Hôpital's:

$$\lim_{x\to 0} \frac{\cos(x)-\sec^2(x)}{3x^2} = \lim_{x\to 0} \frac{-\sin x - 2\sec^2 x\tan x}{6x} = \lim_{x\to 0} \frac{-\cos x - 2(2\sec^2 x\tan^2 x + \sec^4 x)}{6} = -\frac{1}{2}.$$

Hence, the final answer is $2\left(-\frac{1}{2}\right) = -1$. $\blacksquare$

(k) $\lim_{x\to \frac{\pi}{2}^-} \sec(x)^{\cos(x)}$

Solution. Let L be the limit. Then

$$\ln L = \ln\left(\lim_{x\to\frac{\pi}{2}^-} \sec(x)^{\cos(x)}\right) = \lim_{x\to\frac{\pi}{2}^-} \ln\left(\sec(x)^{\cos(x)}\right) = \lim_{x\to(\pi/2)^-} \cos(x)\,\ln(\sec x) = \lim_{x\to(\pi/2)^-} \frac{\ln(\sec x)}{\sec x}.$$

The type is $\frac{\infty}{\infty}$. So by L'Hôpital's:

$$\ln L = \lim_{x\to(\pi/2)^-} \frac{\ln(\sec x)}{\sec x} = \lim_{x\to(\pi/2)^-} \frac{\dfrac{\cancel{\sec x\tan x}}{\sec x}}{\cancel{\sec x\tan x}} = \lim_{x\to(\pi/2)^-} \cos x = 0,$$

and thus $L = 1$. $\blacksquare$

Exc. Find the following limits.

(a) $\lim_{x\to 0} \dfrac{2^x-3^x}{\ln(1-x)}$

(b) $\lim_{x\to 0} \dfrac{\arcsin 2x}{\arcsin 3x}$

(c) $\lim_{x\to 0^+} x^{\,(1-\cos x)\,x}$

20 Linear approximation and differentials

Let f be a function differentiable at $x = a$. Let $L_a(x)$ be the line tangent to the graph of f at $x = a$, viewed as a function of x. Then

$$\frac{L_a(x) - f(a)}{x - a} = f'(a) \quad \Rightarrow \quad L_a(x) = f'(a)(x - a) + f(a).$$

This linear function $L_a(x)$ is called the *linear approximation* of f at a.
The idea is that when x is very close to a, the value of $L_a(x)$ is a good approximation of $f(x)$, even if $f(x)$ itself is hard to compute.

Example. Find the linear approximation of $f(x) = \sqrt[3]{x}$ at a point $x = a$. Use it to approximate $\sqrt[3]{26.9}$ and $\sqrt[3]{1002}$.

Solution. We have $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$. At $x = a$, $f(a) = \sqrt[3]{a}$ and $f'(a) = \dfrac{1}{3\sqrt[3]{a^2}}$. Hence

$$L_a(x) = \sqrt[3]{a} + \frac{1}{3\sqrt[3]{a^2}}(x - a).$$

For $\sqrt[3]{26.9}$, take $a = 27$. Then

$$\sqrt[3]{26.9} \approx L_{27}(26.9) = 3 + \frac{1}{27}(-0.1) = 3 - \frac{1}{270} \approx 2.9963.$$

Note that the actual value of $\sqrt[3]{26.9}$ is $2.996291714\cdots$. So this is an excellent approximation.

For $\sqrt[3]{1002}$, take $a = 1000$. Then

$$\sqrt[3]{1002} \approx L_{1000}(1002) = 10 + \frac{1}{300}(2) = 10.00\overline{6}.$$

Note that the actual value of $\sqrt[3]{1002}$ is $10.00666223\cdots$, again very accurate. ■

We can also use linear approximation to estimate very small changes in $f(x)$ when x changes very slightly. If x changes from a to $a + dx$, where dx is quite tiny, then the small change in $f(x)$ is roughly

$$dy = L_a(a + dx) - L_a(a) = f'(a)\, dx.$$

We call dx and dy the *differentials* of x and y, respectively.

Example. Approximate the change in e^x as x changes from 1 to 1.001.

Solution. Here $f(x) = e^x$, $f'(x) = e^x$. So at $x = 1$, $f'(1) = e$ and $dx = 0.001$. Hence

$$dy \approx f'(1)\, dx = e(0.001) = 0.001e \approx 0.002718.$$

Thus $e^{1.001} \approx e + 0.002718$. ■

Example. Approximate the change in the area of a circle as its radius changes from 3 to 3.001.

Solution. Here $A(r) = \pi r^2$, so $A'(r) = 2\pi r$. At $r = 3$, $A'(3) = 6\pi$, and $dr = 0.001$. Hence

$$dA \approx A'(3)\, dr = 6\pi(0.001) = 0.006\pi \approx 0.01885.$$

So the area increases by approximately 0.01885. ■

21 The sum notation

Given numbers $a_1, a_2, \ldots, a_n$ indexed by $1, 2, \ldots, n$, we use the following notation for "the sum of the a_i's from $i = 1$ to $i = n$."

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Here are some basic properties of the Σ notation:

- For any constant c, we have $\displaystyle\sum_{i=1}^{n} c \;=\; nc$.
- For any constant c, we have $\displaystyle\sum_{i=1}^{n} (c\,a_i) \;=\; c\sum_{i=1}^{n} a_i$.
- We have $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) \;=\; \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$.
- We have $\displaystyle\sum_{i=1}^{n} (a_{i+1} - a_i) \;=\; a_{n+1} - a_1$. This is called a **telescoping sum**.

  Proof. Note that:

  $$\sum_{i=1}^{n} (a_{i+1} - a_i) = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n+1} - a_n) = a_{n+1} - a_1.$$

Example. Prove that for every positive integer n,

$$\sum_{i=1}^{n} i \;=\; \frac{n(n+1)}{2}.$$

Solution. Write the sum forwards and backwards and add:

$$S = \sum_{i=1}^{n} i = 1 + 2 + \cdots + (n-1) + n,$$

$$S = \sum_{i=1}^{n} (n+1-i) = n + (n-1) + \cdots + 2 + 1.$$

Adding up both sides gives

$$2S = \sum_{i=1}^{n} i + \sum_{i=1}^{n} (n+1-i) = \sum_{i=1}^{n} (i+n+1-i) = \sum_{i=1}^{n} (n+1) = n(n+1) \Rightarrow S = \frac{n(n+1)}{2}. \quad \blacksquare$$

Example. Prove that for every positive integer n,

$$\sum_{i=1}^{n} i^2 \;=\; \frac{n(n+1)(2n+1)}{6}.$$

Solution. Recall that $(i+1)^3 = i^3 + 3i^2 + 3i + 1$. So,

$$(i+1)^3 - i^3 \;=\; 3i^2 + 3i + 1.$$

Taking the sum from $i = 1$ to $i = n$, we obtain:

$$\sum_{i=1}^{n} \big((i+1)^3 - i^3\big) = \sum_{i=1}^{n} (3i^2 + 3i + 1) = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1.$$

Now, the left-hand side is a telescoping sum:

$$\sum_{i=1}^{n} \big((i+1)^3 - i^3\big) = (n+1)^3 - 1.$$

For the right-hand side, note that $\sum_{i=1}^{n} 1 = n$, and by the previous example, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Thus

$$3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = 3\sum_{i=1}^{n} i^2 + 3 \cdot \frac{n(n+1)}{2} + n.$$

We deduce that:

$$\begin{aligned}
(n+1)^3 - 1 &= 3\sum_{i=1}^{n} i^2 + \frac{3n(n+1)}{2} + n \\
\Rightarrow 3\sum_{i=1}^{n} i^2 &= (n+1)^3 - 1 - \frac{3n(n+1)}{2} - n = \frac{n(n+1)\big(2n+1\big)}{2} \\
\Rightarrow \sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}. \quad \blacksquare
\end{aligned}$$

Exercise. Prove that for every positive integer n,

$$\sum_{i=1}^{n} i^3 \;=\; \left(\frac{n(n+1)}{2}\right)^2.$$

Exercise[1]. Prove that for every positive integer n,

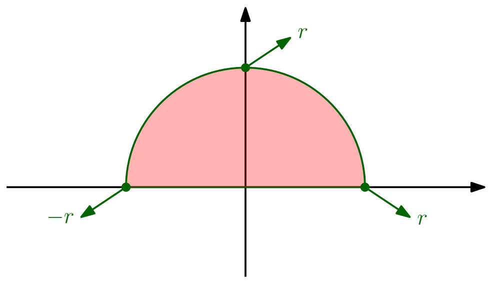
$$\sum_{i=1}^{n} \frac{i}{2^i} \;=\; 2 - \frac{n+2}{2^n}.$$

[1] For very interested students, and beyond the scope of the class!

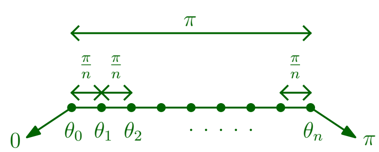
22 Area of a circle

Let's **prove**[2] that the area of a circle of radius r is πr^2.

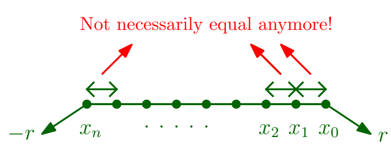
1. Consider the function
$$f(x) = \sqrt{r^2 - x^2}.$$
Note that $D_f = [-r, r]$. Our goal is to prove that the area enclosed between the graph of this function and the x-axis is $\frac{\pi r^2}{2}$.



2. Start with a positive integer n. Divide the interval $[0, \pi]$ into n equal sub-intervals; that is, choose $n+1$ points $\theta_0, \theta_1, \ldots, \theta_n$ in $[0, \pi]$ where
$$\theta_i = \frac{i\pi}{n}, \quad \text{for every } i = 0, 1, \ldots, n.$$
So we have
$$\theta_0 = 0, \quad \theta_1 = \frac{\pi}{n}, \quad \theta_2 = \frac{2\pi}{n}, \quad \ldots, \quad \theta_n = \frac{n\pi}{n} = \pi.$$
Note that for every $i = 1, \ldots, n$, we have $\theta_i - \theta_{i-1} = \dfrac{\pi}{n}$.



3. For every $i = 0, 1, \ldots, n$, define
$$x_i = r\cos\theta_i.$$
So we have
$$x_0 = r\cos 0 = r, \quad x_1 = r\cos\left(\frac{\pi}{n}\right), \quad x_2 = r\cos\left(\frac{2\pi}{n}\right), \quad \ldots, \quad x_n = r\cos(\pi) = -r.$$
It follows that $x_0, x_1, \ldots, x_n$ break $[-r, r]$ into n (not necessarily equal) sub-intervals.



[2]This example is for demonstration only, and you will *not* be asked to solve problems at this level of difficulty.

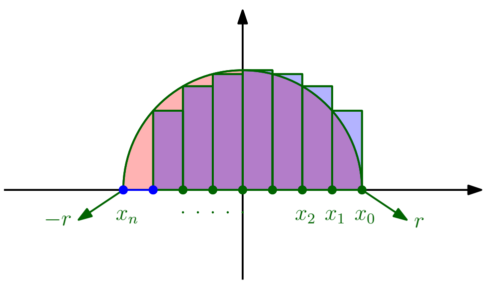
4. For every $i = 1, \ldots, n$, consider the rectangle with vertices

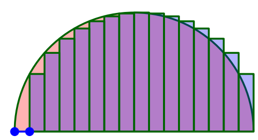
$$(x_{i-1}, 0), (x_i, 0), (x_i, f(x_i)), (x_{i-1}, f(x_i)).$$

Let S_i be the area of this rectangle. Then we have

$$S_i = f(x_i)(x_{i-1} - x_i).$$

The **key observation** is that the sum of $S_1, \ldots, S_n$ approximates the area of the semicircle, and, **the bigger the choice of n we started with, the better the approximation!**





5. We have a new notation for taking the sum of a bunch of variables that all depend on an integer parameter:

$$\sum_{i=1}^{n} S_i = S_1 + S_2 + \cdots + S_n.$$

Using this notation, what we said above can be rewritten mathematically as follows:

$$\text{The area of the semicircle} = \lim_{n\to\infty} \sum_{i=1}^{n} S_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)(x_{i-1} - x_i).$$

6. Now, for every $i = 1, \ldots, n$, recall that by the definition of f and the choice of x_i and x_{i-1}, we have:

$$S_i = f(x_i)(x_{i-1} - x_i) = \sqrt{r^2 - r^2\cos^2\theta_i}\cdot(r\cos\theta_{i-1} - r\cos\theta_i) = r^2(\sin\theta_i\cos\theta_{i-1} - \sin\theta_i\cos\theta_i).$$

Since

$$\theta_i = \frac{i\pi}{n}, \quad \theta_{i-1} = \frac{(i-1)\pi}{n},$$

using the sum/difference identity, we have

$$\sin\theta_i\cos\theta_{i-1} = \frac{1}{2}(\sin(\theta_i + \theta_{i-1}) + \sin(\theta_i - \theta_{i-1})) = \frac{1}{2}\sin\left(\frac{(2i-1)\pi}{n}\right) + \frac{1}{2}\sin\left(\frac{\pi}{n}\right);$$

and using the double-angle identity, we have

$$\sin\theta_i\cos\theta_i = \frac{1}{2}\sin(2\theta_i) = \frac{1}{2}\sin\left(\frac{2i\pi}{n}\right).$$

Thus, substituting blue for blue and red for red, we have

$$\sin\theta_i\cos\theta_{i-1} - \sin\theta_i\cos\theta_i = \frac{1}{2}\sin\left(\frac{(2i-1)\pi}{n}\right) + \frac{1}{2}\sin\left(\frac{\pi}{n}\right) - \frac{1}{2}\sin\left(\frac{2i\pi}{n}\right).$$

It follows that

$$\begin{aligned}
\sum_{i=1}^{n} S_i &= r^2 \sum_{i=1}^{n} (\sin\theta_i \cos\theta_{i-1} - \sin\theta_i\cos\theta_i) \\
&= \frac{r^2}{2}\sum_{i=1}^{n}\left(\sin\left(\frac{(2i-1)\pi}{n}\right) + \sin\left(\frac{\pi}{n}\right) - \sin\left(\frac{2i\pi}{n}\right)\right) \\
&= \frac{r^2}{2}\left(\underbrace{\sum_{i=1}^{n}\sin\left(\frac{(2i-1)\pi}{n}\right)}_{\text{First sum}} + \underbrace{\sum_{i=1}^{n}\sin\left(\frac{\pi}{n}\right)}_{\text{Second sum}} - \underbrace{\sum_{i=1}^{n}\sin\left(\frac{2i\pi}{n}\right)}_{\text{Third sum}}\right)
\end{aligned}$$

7. Note that the first sum is the "sum of sines of the odd multiples of $\frac{\pi}{n}$ from 1 to $2n-1$", and the third sum is the "sum of sines of the even multiples of $\frac{\pi}{n}$ from 2 to $2n$. Thus, together, they are the "sum of sines of all multiples of $\frac{\pi}{n}$ from 1 to $2n$"; that is:

$$\begin{aligned}
\text{First sum} + \text{Third sum} = \sum_{i=1}^{2n}\sin\left(\frac{i\pi}{n}\right) &= \sum_{i=1}^{n}\sin\left(\frac{i\pi}{n}\right) + \sum_{i=n+1}^{2n}\sin\left(\frac{i\pi}{n}\right) \\
&= \sum_{i=1}^{n}\sin\left(\frac{i\pi}{n}\right) + \sum_{i=1}^{n}\sin\underbrace{\left(\frac{(n+i)\pi}{n}\right)}_{\pi+\frac{i\pi}{n}} \\
&= \sum_{i=1}^{n}\left(\sin\left(\frac{i\pi}{n}\right) + \sin\left(\pi + \frac{i\pi}{n}\right)\right) \\
&= \sum_{i=1}^{n}\left(\sin\left(\frac{i\pi}{n}\right) - \sin\left(\frac{i\pi}{n}\right)\right) = 0.
\end{aligned}$$

For the second sum, note that $\sin\left(\frac{\pi}{n}\right)$ is a "constant" (meaning it doesn't vary as i changes). So,

$$\text{Second sum} = \sum_{i=1}^{n}\sin\left(\frac{\pi}{n}\right) = n\sin\left(\frac{\pi}{n}\right).$$

8. Putting the three sums back together, we have:

$$\sum_{i=1}^{n} S_i = \frac{r^2}{2}\cdot n\sin\left(\frac{\pi}{n}\right) = \frac{r^2}{2}\cdot\frac{\pi\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = \frac{\pi r^2}{2}\cdot\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}};$$

Recall that:

$$\text{The area of the semicircle} = \lim_{n\to\infty}\sum_{i=1}^{n} S_i = \lim_{n\to\infty}\frac{\pi r^2}{2}\cdot\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = \frac{\pi r^2}{2}\cdot\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}.$$

Also, as $n\to\infty$, we have $\frac{\pi}{n}\to 0$, and so $\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} = 1$. Hence, we proved that

$$\text{The area of the semicircle} = \frac{\pi r^2}{2}. \quad \blacksquare$$

Here are two intriguing questions to take away:

- Recall that we found the area enclosed between the graph of $f(x) = \sqrt{r^2 - x^2}$ and the x-axis over $[-r, r]$. But why don't we do this more generally? In other words, why would we not think about **the area enclosed between the graph of a function f and the x-axis over a closed interval $[a, b]$ in the domain of f?**

- We just saw that the area of a circle, as a function of its radius r, is

  $$f(r) = \pi r^2.$$

  If we "go to the boundary", the perimeter of the circle is

  $$g(r) = 2\pi r.$$

  You probably also know that the volume of a sphere, as a function of its radius r, is

  $$f(r) = \frac{4}{3}\pi r^3.$$

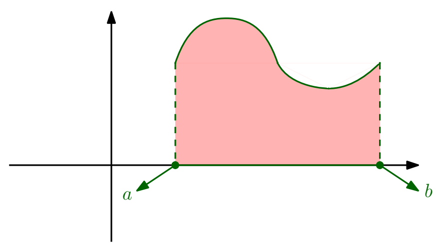
  If we "go to the boundary", the surface area of the sphere is

  $$g(r) = 4\pi r^2.$$

  ... wait, g is always the derivative of f! Is that random?!

23 Definite integrals

Let f be a function defined over a closed interval $[a, b]$ and suppose that $f(x) \geq 0$ for all $x \in [a, b]$. Our goal is to define the area under the graph of f and above the x–axis between a and b, through the following steps.



1. Fix a positive integer n. Divide $[a, b]$ into n equal sub-intervals by choosing points

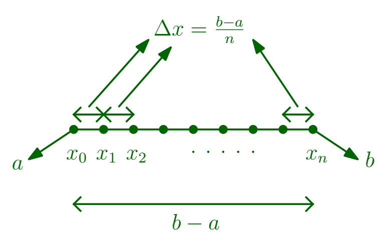
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

where $x_i = a + i\left(\frac{b-a}{n}\right)$. So $x_0 = a$, $x_n = b$, and for every $i = 1, \dots, n$, we have

$$x_i - x_{i-1} = \frac{b-a}{n}.$$

We denote this quantity, the equal length of all sub-intervals, by Δx. So

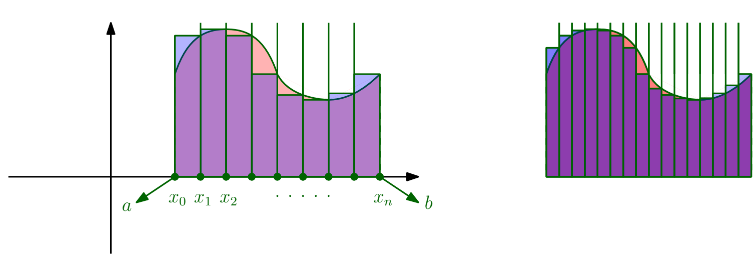
$$\Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{for all} \quad i = 0, 1, \dots, n.$$



2. On each subinterval $[x_{i-1}, x_i]$, take the rectangle of height $f(x_i)$, which is **the value of f at the right endpoint**. The area of that rectangle is $f(x_i)(x_i - x_{i-1}) = f(x_i)\,\Delta x$. The sum of these areas is called the *right Riemann sum* of f from a to b:

$$\text{The right Riemann sum} = \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

3. The key observation is that the right Riemann sum is a good estimate of the area we would like to compute. In fact, **the more rectangles the better the estimate**, so that as $n \to \infty$, the Riemann sum converges to the area under the graph of f on $[a, b]$!



We call this limit *the definite integral of f from a to b*, and denote it by $\int_a^b f(x)\,dx$.

$$\boxed{\begin{array}{c} \displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x \\[2ex] \text{where} \quad \Delta x = \dfrac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{for all} \quad i = 0,1,\dots,n. \end{array}}$$

We call a the *lower limit*, and we call b the *upper limit*, and we call $f(x)$ the *integrand*.

Some remarks:

- We could instead choose $f(x_{i-1})$, the value of f at the left endpoint of $[x_{i-1}, x_i]$, to be the height of the rectangle on top. That would yield the "left Riemann sum." Indeed, we could even choose $f(x^*)$ for any point $x^* \in [x_{i-1}, x_i]$ to be the height of the rectangle on top. One may show that all such Riemann sums converge to the same limit. So, in this course, **we always stick with the right Riemann sum**.
- In general, if $f(x)$ is sometimes negative over $[a, b]$, then

  $$\boxed{\int_a^b f(x)\,dx = (\text{area under } f \text{ and above the } x\text{-axis}) - (\text{area above } f \text{ and under the } x\text{-axis}).}$$

  In other words, the definite integral actually represents the **signed** area under f from a to b.
- It follows immediately from the geometric interpretation of signed area that:

  $$\boxed{\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.}$$

  In particular, we have $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0$, and thus

  $$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.}$$

Example. Use the definition of definite integral to evaluate $\displaystyle\int_0^4 x^3\,dx$.

Solution. We have

$$\int_0^4 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^n x_i^3\,\Delta x$$

$$\text{where} \quad \Delta x = \frac{4-0}{n} = \frac{4}{n} \quad \text{and} \quad x_i = 0 + i\Big(\frac{4}{n}\Big) = \frac{4i}{n} \quad \text{for all} \quad i = 0,1,\dots,n.$$

So,

$$\int_0^4 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^n x_i^3\,\Delta x = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n\left(\frac{4i}{n}\right)^3 = \lim_{n\to\infty}\frac{4^4}{n^4}\sum_{i=1}^n i^3.$$

Recall that

$$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}.$$

Thus,

$$\int_0^4 x^3\,dx = \lim_{n\to\infty}\frac{4^4}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \lim_{n\to\infty}\frac{64(n^2+2n+1)}{n^2} = \lim_{n\to\infty}\frac{64\cancel{n^2}\left(1+\cancelto{0}{\frac{2}{n}+\frac{1}{n^2}}\right)}{\cancel{n^2}} = 64. \quad \blacksquare$$

Example. Use the definition of definite integral to evaluate $\displaystyle\int_{-1}^2 (3+2x)\,dx$.

Solution. We have

$$\int_{-1}^2 (3+2x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n (3+2x_i)\,\Delta x$$

$$\text{where}\quad \Delta x = \frac{3}{n}\quad\text{and}\quad x_i = -1+\frac{3i}{n}\quad\text{for all}\quad i = 0,1,\ldots,n.$$

So,

$$\int_{-1}^2 (3+2x)\,dx = \lim_{n\to\infty}\sum_{i=1}^n (3+2x_i)\,\Delta x = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n\left(3+2\left(-1+\frac{3i}{n}\right)\right) = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n\left(1+\frac{6i}{n}\right).$$

Now, recall that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$, and thus:

$$\sum_{i=1}^n\left(1+\frac{6i}{n}\right) = \sum_{i=1}^n 1 + \sum_{i=1}^n\frac{6i}{n} = \sum_{i=1}^n 1 + \frac{6}{n}\sum_{i=1}^n i = n + 3(n+1) = 4n+3.$$

Therefore,

$$\int_{-1}^2 (3+2x)\,dx = \lim_{n\to\infty}\frac{3}{n}\cdot(4n+3) = \lim_{n\to\infty}\frac{12n+9}{n} = \cdots = 12. \quad \blacksquare$$

Exercise. Use the definition of definite integral to evaluate $\displaystyle\int_0^2 (x^2+x+1)\,dx$.

24 Antiderivatives and the Fundamental Theorem of Calculus

An *antiderivative* of a function $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

For instance, $\dfrac{x^4}{4}$ is an antiderivative of x^3, and $-\cos x$ is an antiderivative of $\sin x$, because:

$$\left(\frac{x^4}{4}\right)' = x^3 \quad \text{and} \quad (-\cos x)' = \sin x.$$

What does this have to do with definite integrals?
Answer. Recall the definition of the definite integral of a function f from a to b:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$\text{where} \quad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x \quad \text{for all} \quad i = 0, 1, \ldots, n.$$

As we saw in a few examples, directly evaluating this limit is often painfully difficult (and for some integrands almost impossible – try evaluating $\int_1^2 \sqrt{x}\,dx$ using only Riemann sums before reading further!).

Luckily, there is a remarkable theorem that helps work around this issue:

Fundamental Theorem of Calculus (Part 1). *Let $f(x)$ be continuous on $[a, b]$, and let $F(x)$ be any antiderivative of f, meaning $F'(x) = f(x)$. Then*

$$\int_a^b f(x)\,dx = \underbrace{[F(x)]_a^b}_{\text{Notation for } F(b)-F(a)} = F(b) - F(a).$$

We will not prove this theorem, but the intuition is the following: Recall that for $y = F(x)$, the differential relation gives

$$dy = F'(x)\,dx.$$

Since $F'(x) = f(x)$, we have $dy = f(x)\,dx$. In particular,

$$\int_a^b f(x)\,dx = \int_a^b dy,$$

which represents the "sum of all the tiny changes in $F(x)$ as x moves from a to b." But this accumulated change should ultimately be exactly the net change in the value of F between $x = a$ and $x = b$, that is $F(b) - F(a)$!

Now, let's take a look back at two previous examples.

Example. Use FTC1 to compute $\displaystyle\int_0^4 x^3\,dx$.

Solution. An antiderivative of $f(x) = x^3$ is $F(x) = \frac{x^4}{4}$. So,

$$\int_0^4 x^3\,dx = \left[\frac{x^4}{4}\right]_0^4 = \frac{4^4}{4} - \frac{0^4}{4} = 64. \quad \blacksquare$$

Example. Use FTC1 to compute $\displaystyle\int_{-1}^{2} (3+2x)\,dx$.

Solution. An antiderivative of $f(x) = 3 + 2x$ is $F(x) = 3x + x^2$. So

$$\int_{-1}^{2} (3+2x)\,dx = \left[3x + x^2\right]_{-1}^{2} = (3(2)+2^2)-(3(-1)+(-1)^2) = (6+4)-(-3+1) = 10-(-2) = 12. \quad \blacksquare$$

In light of FTC1, it is clear that antiderivatives make our lives much easier when computing definite integrals. But do we always have this privilege? Specifically:

Does every continuous function even have an antiderivative?

Yes! And this comes from the *second* part of the Fundamental Theorem.

Given a function f, consider a definite integral made in the following way:

1. The integrand is f.
2. The lower limit is some fixed number a.
3. **The upper limit is a variable; call it $\boldsymbol{x}$.**

Since the upper limit is now x, to avoid confusion we write f as a function of something else, let's say t. So we are thinking about

$$\int_a^x f(t)\,dt.$$

Here is the catch:

This definite integral is itself a ***function of $\boldsymbol{x}$***.

The second part of the Fundamental Theorem says that this function is an antiderivative of f.

Fundamental Theorem of Calculus (Part 2). *Let f be a continuous function. Fix $a \in \mathbb{R}$ and define*

$$F(x) = \underbrace{\int_a^x f(t)\,dt}_{\text{Function of } x}.$$

Then F is an antiderivative of f; that is,

$$F'(x) = f(x).$$

In other words,

$$\frac{d}{dx}\underbrace{\int_a^x f(t)\,dt}_{\text{Function of } x} = f(x).$$

Examples

Example. Compute $\dfrac{d}{dx}\displaystyle\int_1^x t^3\,dt$.

Solution. Note that $\frac{t^4}{4}$ is an antiderivative of t^3. So, using FTC1,

$$\int_1^x t^3\,dt = \left[\frac{t^4}{4}\right]_1^x = \frac{x^4}{4} - \frac{1}{4},$$

and differentiating gives

$$\frac{d}{dx}\int_1^x t^3\,dt = \frac{d}{dx}\left(\frac{x^4}{4} - \frac{1}{4}\right) = x^3.$$

Alternatively, using FTC2, the answer is immediate:

$$\frac{d}{dx}\int_1^x t^3\,dt = x^3. \quad \blacksquare$$

Example. Compute $\dfrac{d}{dx}\displaystyle\int_{-1}^x \sin\bigl(\cos(e^{\cos(\sin(t))})\bigr)\,dt$.

Solution. Nobody knows an antiderivative of $\sin\bigl(\cos(e^{\cos(\sin(t))})\bigr)$, and so FTC1 is useless here! But FTC2 gives:

$$\frac{d}{dx}\int_{-1}^x \sin(\cos(e^{\cos(\sin(t))}))\,dt = \sin(\cos(e^{\cos(\sin(x))})).\blacksquare$$

Example. Compute $\dfrac{d}{dx}\displaystyle\int_2^{1-\cos x} \sqrt{1-(1-t)^2}\,dt$.

Solution. This is a composition. Define

$$F(x) = \int_2^x \sqrt{1-(1-t)^2}\,dt.$$

Then

$$\int_2^{1-\cos x} \sqrt{1-(1-t)^2}\,dt = F(1-\cos x).$$

And so by the chain rule:

$$\frac{d}{dx}\int_2^{1-\cos x} \sqrt{1-(1-t)^2}\,dt = (1-\cos x)' \cdot F'(1-\cos x) = \sin x \cdot F'(1-\cos x).$$

On the other hand, by FTC2,

$$F'(x) = \frac{d}{dx}\int_2^x \sqrt{1-(1-t)^2}\,dt = \sqrt{1-(1-x)^2}.$$

Therefore,

$$\frac{d}{dx}\int_2^{1-\cos x} \sqrt{1-(1-t)^2}\,dt = \sin x \cdot \sqrt{1-(1-(1-\cos x))^2} = \sin x\,|\sin x|. \quad \blacksquare$$

In general, using the same proof, we can show that:

If g is differentiable and f is continuous, then

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = g'(x)f(g(x)).$$

Example. Compute $\displaystyle\frac{d}{dx}\int_{\cos x}^{\sin x} e^{t^2}\,dt$.

Solution. We can "break" the definite integral at 0 (or any other number):

$$\int_{\cos x}^{\sin x} e^{t^2}\,dt = \int_0^{\sin x} e^{t^2}\,dt + \int_{\cos x}^{0} e^{t^2}\,dt = \int_0^{\sin x} e^{t^2}\,dt - \int_0^{\cos x} e^{t^2}\,dt.$$

Now, let

$$F(x) = \int_0^x e^{t^2}\,dt.$$

Then

$$\int_{\cos x}^{\sin x} e^{t^2}\,dt = \int_0^{\sin x} e^{t^2}\,dt - \int_0^{\cos x} e^{t^2}\,dt = F(\sin x) - F(\cos x).$$

Thus, by the chain rule:

$$\frac{d}{dx}\int_{\cos x}^{\sin x} e^{t^2}\,dt = (\sin x)' \cdot F'(\sin x) - (\cos x)' \cdot F'(\cos x) = \cos x \cdot F'(\sin x) + \sin x \cdot F'(\cos x).$$

On the other hand, by FTC2, we have

$$F'(x) = \frac{d}{dx}\int_0^x e^{t^2}\,dt = e^{x^2}.$$

Therefore,

$$\frac{d}{dx}\int_{\cos x}^{\sin x} e^{t^2}\,dt = e^{\sin^2 x}\cos x + e^{\cos^2 x}\sin x. \quad \blacksquare$$

In general, using the same proof, we can show that:

If g and h are differentiable and f is continuous, then

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt = g'(x)f(g(x)) - h'(x)f(h(x)).$$

A summary:

- An antiderivative is the inverse of the derivative.
- FTC1 tells us that definite integrals can be evaluated using antiderivatives: If $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- FTC2 tells us that *every* continuous function f has an antiderivative F, as follows:

$$F(x) = \int_0^x f(t)\,dt.$$

Sometimes this antiderivative is easy to find. For $f(x) = x^3$, by FTC1, we have

$$F(x) = \int_0^x t^3\,dt = \frac{x^4}{4} - \frac{0^4}{4} = \frac{x^4}{4},$$

as it should!

Sometimes it is impossible to express the antiderivative in elementary terms. For example, $f(x) = \sin\left(\frac{1}{x}\right)$ **does** have an antiderivative (by FTC2), but this antiderivative has no elementary closed form.

Sometimes an antiderivative exists and can be found, but requires nontrivial techniques. Next, we will learn methods that allow us to find pretty complicated antiderivatives; for instance, that of

$$f(x) = \frac{1}{\sin x + \cos x},$$

which looks hopeless at first!

25 Integration techniques

Recall that FTC1 allows us to use antiderivatives to evaluate definite integrals, and FTC2 tells us that antiderivatives always exist. But how can we find an antiderivative of a given function $f(x)$?

First, notice that we keep saying **an** antiderivative. But why? Are there many?
Yes – For example, the following are all antiderivatives of $f(x) = x^3$:

$$\frac{x^4}{4}, \quad \frac{x^4}{4} - 1, \quad \frac{x^4}{4} + \sqrt[17]{\pi}.$$

In fact, one can prove that:

Theorem. If $F(x)$ is *an* antiderivative of $f(x)$, then *every* antiderivative of f is of the form $F(x)+C$ for some constant C.

The collection of all antiderivatives of $f(x)$ is called the *indefinite integral of* f, and is denoted by

$$\int f(x)\,dx = F(x) + C.$$

From here on, we are going to learn several techniques to find indefinite integrals; that is, to find an antiderivative of a function f, and then put a "$+C$" next to it to declare that "we have found all possible antiderivatives of the function f."

25.1 Basic properties and familiar antiderivatives

Here is a good set of tools to get us started:

- $\displaystyle\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$
- $\displaystyle\int k\,f(x)\,dx = k\int f(x)\,dx$, where k is a constant.
- **The power rule for integrals**: $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{when} \quad n \neq -1.$
- What if $n = -1$? Then: $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C.$

  Note. We are putting $|x|$ instead of x because $\frac{1}{x}$ is defined everywhere, but $\ln x$ is only defined where $x > 0$. This is our workaround to extend ln to be defined everywhere (except at zero).
- Some integrals using trigonometric functions and their inverses:

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C,$$

$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \sec x\,\tan x\,dx = \sec x + C,$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C, \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C,$$

- Exponentials: $\displaystyle\int e^x\,dx = e^x + C \quad \text{and} \quad \int a^x\,dx = \frac{a^x}{\ln(a)} + C \quad (\text{for } a > 0, a \neq 1).$

Example. Compute each of the following definite or indefinite integrals.

(a) $\displaystyle\int \left(13x^6 - x\right)\, dx.$

Solution. $\displaystyle\int \left(13x^6 - x\right)\, dx = \frac{13x^7}{7} - \frac{x^2}{2} + C.$ ■

(b) $\displaystyle\int \left(\pi^x - x^\pi - \cos x\right)\, dx.$

Solution. $\displaystyle\int \left(\pi^x - x^\pi - \cos x\right)\, dx = \frac{\pi^x}{\ln \pi} - \frac{x^{\pi+1}}{\pi+1} - \sin x + C.$ ■

(c) $\displaystyle\int \left(\sqrt{x} - \sqrt[3]{x^2} + \sqrt[4]{x^3}\right)\, dx.$

Solution. We have

$$\int \left(\sqrt{x} - \sqrt[3]{x^2} + \sqrt[4]{x^3}\right)\, dx = \int \left(x^{1/2} - x^{2/3} + x^{3/4}\right)\, dx = \frac{2}{3}x^{3/2} - \frac{3}{5}x^{5/3} + \frac{4}{7}x^{7/4} + C. \quad ■$$

(d) $\displaystyle\int (x^5 - \sqrt{x})^2\, dx.$

Solution. We have

$$\int (x^5 - \sqrt{x})^2\, dx = \int \left(x^{10} - 2x^{11/2} + x\right)\, dx = \frac{x^{11}}{11} - \frac{4}{13}x^{13/2} + \frac{x^2}{2} + C. \quad ■$$

(e) $\displaystyle\int \frac{1 - 3x^2}{1 + x^2}\, dx.$

Solution. We have

$$\int \frac{1 - 3x^2}{1 + x^2}\, dx = \int \frac{4 - 3(1 + x^2)}{1 + x^2}\, dx = \int \left(\frac{4}{1 + x^2} - 3\right)\, dx = 4\tan^{-1} x - 3x + C. \quad ■$$

(f) $\displaystyle\int \frac{\sqrt[5]{x} - 5\sqrt{x}}{x\sqrt{x}}\, dx.$

Solution. We have

$$\int \frac{\sqrt[5]{x} - 5\sqrt{x}}{x\sqrt{x}}\, dx = \int \frac{x^{1/5} - 5x^{1/2}}{x^{3/2}}\, dx = \int \left(x^{-4/5} - 5x^{-1}\right)\, dx = \frac{5}{1}x^{1/5} - 5\ln|x| + C.$$

(g) $\displaystyle\int \left(2^{3x} - 3^{2x}\right)\, dx.$

Solution. We have

$$\int \left(2^{3x} - 3^{2x}\right)\, dx = \frac{2^{3x}}{3\ln 2} - \frac{3^{2x}}{2\ln 3} + C.$$

Notice how 3 and 2 are incorporated so that when we take the derivative of the right hand side, things cancel out together and we get to the left hand side! (More on this when we introduce substitution.) ■

(h) $\displaystyle\int \sin^2 x\, dx.$

Solution. Recall that $\sin^2 x = \dfrac{1-\cos 2x}{2}$. So,

$$\int \sin^2 x\, dx = \frac{1}{2}\int (1-\cos 2x)\, dx = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Again, notice how 2 comes to play. ■

(i) $\displaystyle\int_{-2}^{2} |3x-4|\, dx.$

Solution. Note that $|3x-4| = \begin{cases} 3x-4, & x \geq \frac{4}{3}, \\ 4-3x, & x < \frac{4}{3}. \end{cases}$. So break the integral at $\frac{4}{3}$:

$$\int_{-2}^{2} |3x-4|\, dx = \underbrace{\int_{-2}^{4/3} (4-3x)\, dx}_{A} + \underbrace{\int_{4/3}^{2} (3x-4)\, dx}_{B}.$$

Now, $\displaystyle\int (4-3x)\, dx = 4x - \frac{3x^2}{2} + C_1$ and $\displaystyle\int (3x-4)\, dx = \frac{3x^2}{2} - 4x + C_2$. Therefore, by FTC1, we have

$$A = \left(4(4/3) - \frac{3(4/3)^2}{2} + \cancel{C_1}\right) - \left(4(-2) - \frac{3(-2)^2}{2} + \cancel{C_1}\right) = \left(\frac{16}{3} - \frac{8}{3}\right) - (-8-6) = \frac{50}{3}.$$

and

$$B = \left(\frac{3(2)^2}{2} - 4(2) + \cancel{C_2}\right) - \left(\frac{3(4/3)^2}{2} - 4(4/3) + \cancel{C_2}\right) = (6-8) - \left(\frac{8}{3} - \frac{16}{3}\right) = \frac{2}{3}.$$

Hence, $\displaystyle\int_{-2}^{2} |3x-4|\, dx = A + B = \frac{52}{3}.$ ■

Note. We could have ignored C_1 and C_2. In general, when an indefinite integral is to be used (as in FTC1) for evaluating a definite integral, we can ignore the "$+C$" thingy.

(j) $\displaystyle\int_{-1}^{3} f(x)\, dx$, where $f(x) = \begin{cases} 1-x, & x > 1, \\ x^2-1, & x < 1. \end{cases}$.

Solution. Breaking the integral at 1, we obtain $\displaystyle\int_{-1}^{3} f(x)\, dx = \underbrace{\int_{-1}^{1} (x^2-1)\, dx}_{A} + \underbrace{\int_{1}^{3} (1-x)\, dx}_{B}.$

Now, $\displaystyle\int (x^2-1)\, dx = \frac{x^3}{3} - x + C_1$ and $\displaystyle\int (1-x)\, dx = x - \frac{x^2}{2} + C_2$. Therefore, by FTC1, and ignoring C_1 and C_2, we have:

$$A = \left[\frac{x^3}{3} - x\right]_{-1}^{1} = \left(\frac{1}{3} - 1\right) - \left(-\frac{1}{3} + 1\right) = -\frac{2}{3} - \frac{2}{3} = -\frac{4}{3}.$$

and

$$B = \left[x - \frac{x^2}{2}\right]_{1}^{3} = \left(3 - \frac{9}{2}\right) - \left(1 - \frac{1}{2}\right) = -\frac{3}{2} - \frac{1}{2} = -2.$$

Hence, $\displaystyle\int_{-1}^{3} f(x)\, dx = A + B = -\frac{10}{3}.$ ■

25.2 Substitution

This is essentially the reverse of the chain rule. We use it when:

- the integrand involves a composition of two functions, and
- the derivative of the inner function is also "kinda" there (e.g. up to a multiplicative factor).

In other words, the integrand has the form of "something around" $g'(x)f(g(x))$. In this case, we apply the following steps:

1. Call the inner function $u = g(x)$. Then differentiate with respect to x:
$$\frac{du}{dx} = g'(x) \Rightarrow \boxed{dx = \frac{du}{g'(x)}.}$$
2. Substitute $g(x)$ with u and dx with $\dfrac{du}{g'(x)}$. For instance,
$$\int g'(x)f(g(x))\,dx = \int f(u)\,du.$$
3. Integrate with respect to u, and finally substitute $u = g(x)$ back.

For example, to evaluate $\displaystyle\int x(1-x^2)^{13}\,dx$, notice that

- the integrand involves $(1-x^2)^{13}$, which is a composition $(x^{13}) \circ (1-x^2)$; and
- the derivative of the inner function $(1-x^2)' = -2x$ is also "kinda" there.

So following the above steps would be:

1. Let $u = 1-x^2$. Then $\dfrac{du}{dx} = -2x$, so $dx = -\dfrac{du}{2x}$.
2. Substitute $(1-x^2)$ with u and dx with $-\dfrac{du}{2x}$:
$$\int x(1-x^2)^{13}\,dx = \int xu^{13} \cdot \left(-\frac{du}{2x}\right) = -\frac{1}{2}\int u^{13}\,du$$
3. Integrate with respect to u, and then put $u = g(x)$ back in:
$$\int x(1-x^2)^{13}\,dx = -\frac{1}{2}\int u^{13}\,du = -\frac{1}{28}u^{14} + C = -\frac{1}{28}(1-x^2)^{14} + C.$$

A few more examples:

Example. Evaluate the following integrals using substitution.

(a) $\displaystyle\int x^3\sqrt[6]{2x^4-9}\,dx.$

Solution. Let $u = 2x^4 - 9$. Then $dx = du/(8x^3)$. So,
$$\int x^3\sqrt[6]{2x^4-9}\,dx = \int x^3u^{1/6}\,\frac{du}{8x^3} = \frac{1}{8}\int u^{1/6}\,du = \frac{1}{8}\cdot\frac{6}{7}u^{7/6} + C = \frac{3}{28}(2x^4-9)^{7/6} + C. \quad \blacksquare$$

(b) $\displaystyle\int \frac{x}{\sqrt{1-2x}}\,dx.$

Solution. Let $u = 1-2x$. Then, $dx = du/(-2)$. Also, we have $x = \frac{1-u}{2}$. Thus,

$$\int \frac{x}{\sqrt{1-2x}}\,dx = -\frac{1}{4}\int \frac{1-u}{\sqrt{u}}\,du = -\frac{1}{4}\int (u^{-1/2} - u^{1/2})\,du = -\frac{1}{4}\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= -\frac{1}{2}\sqrt{1-2x} + \frac{1}{6}(1-2x)^{3/2} + C. \quad \blacksquare$$

(c) $\displaystyle\int \tan x\,dx.$

Solution. Note that $\tan x = \dfrac{\sin x}{\cos x}$. Let $u = \cos x$. Then $dx = -\dfrac{du}{\sin x}$. So,

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x} = \int \frac{\sin x}{u}\cdot\left(-\frac{du}{\sin x}\right) = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C. \quad \blacksquare$$

(d) $\displaystyle\int \sec^7 x \tan x\,dx.$

Solution. Let $u = \sec x$. Then $du = \sec x \tan x\,dx$, so $dx = \dfrac{du}{\sec x \tan x} = \dfrac{du}{u \tan x}$. So,

$$\int \sec^7 x \tan x\,dx = \int u^7 \tan x \cdot \left(\frac{du}{u\tan x}\right) = \int u^6\,du = \frac{u^7}{7} + C = \frac{\sec^7 x}{7} + C. \quad \blacksquare$$

(e) $\displaystyle\int \frac{x^8}{\sqrt[5]{4-x^3}}\,dx.$

Solution. Let $u = 4 - x^3$. Then $dx = -\dfrac{du}{3x^2}$. Thus,

$$\int \frac{x^8}{(4-x^3)^{1/5}}\,dx = \int \frac{x^8}{u^{1/5}}\cdot\left(\frac{du}{3x^2}\right) = -\frac{1}{3}\int \frac{x^6}{u^{1/5}}\,du.$$

Also, note that $x^6 = (x^3)^2 = (4-(4-x^3))^2 = (4-u)^2 = u^2 - 8u + 16$. So,

$$\int \frac{x^8}{(4-x^3)^{1/5}}\,dx = -\frac{1}{3}\int \frac{u^2-8u+16}{u^{1/5}}\,du = -\frac{1}{3}\int \left(u^{9/5} - 8u^{4/5} + 16u^{-1/5}\right)du$$

$$= -\frac{1}{3}\left(\frac{5}{14}u^{14/5} - \frac{40}{9}u^{9/5} + 20u^{4/5}\right) + C$$

$$= -\frac{1}{3}\left(\frac{5}{14}(4-x^3)^{14/5} - \frac{40}{9}(4-x^3)^{9/5} + 20(4-x^3)^{4/5}\right) + C. \quad \blacksquare$$

(f) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx.$

Solution. Let $u = \sqrt{x}$. Then $dx = 2\sqrt{x}\,du = 2u\,du$. So,

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^u}{u}\cdot(2u\,du) = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C. \quad \blacksquare$$

(g) $\displaystyle\int_{e^2}^{e^3} \frac{\ln x}{x}\, dx.$

Solution. Let $u = \ln x$, so $dx = x du$. Then

$$\int \frac{\ln x}{x}\, dx = \int \frac{u}{x} \cdot (x du) = \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C.$$

Therefore, by FTC1,

$$\int_{e^2}^{e^3} \frac{\ln x}{x}\, dx = \frac{1}{2}\Big[(\ln x)^2\Big]_{e^2}^{e^3} = \frac{1}{2}(9-4) = \frac{5}{2}. \quad \blacksquare$$

Note. Like we saw above, for evaluating definite integrals using substitution, it would be the safest to first finish finding the antiderivative – that is, get to an **answer in terms of $\boldsymbol{x}$** – and then apply FTC1 and the upper/lower limits.

Alternatively, *you could* apply FTC1 to the intermediate integral in terms of u, but in order to do that, **you must figure the upper/lower limits for u right away using $\boldsymbol{u=g(x)}$ and keep everything in terms of $\boldsymbol{u}$.**

For instance, here, we have $u = \ln x$. So, when $x = e^2$, we have $u = 2$, and when $x = e^3$, we have $u = 3$. Hence,

$$\int_{e^2}^{e^3} \frac{\ln x}{x}\, dx = \int_2^3 u\, du = \frac{1}{2}\Big[u^2\Big]_2^3 = \frac{1}{2}(9-4) = \frac{5}{2}.$$

(Though I still recommend taking the first approach!)

(h) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3(2x)\, dx.$

Solution. By the double-angle identity, $\sin^3(2x) = 8\sin^3 x\cos^3 x$. Let $u = \sin x$. Then $du = \cos x\, dx$, so $dx = \dfrac{du}{\cos x}$. So

$$\int \sin^3(2x)\, dx = 8\int \sin^3 x\cos^3 x\, dx = 8\int u^3\cos^3 x \cdot \left(\frac{du}{\cos x}\right) = 8\int u^3 \underbrace{\cos^2 x}_{1-u^2}\, du$$

$$= 8\int (u^3 - u^5)\, du = 8\left(\frac{u^4}{4} - \frac{u^6}{6}\right) + C = 2\sin^4 x - \frac{4}{3}\sin^6 x + C$$

$$\int_0^{\frac{\pi}{2}} \sin^3(2x)\, dx = \frac{1}{2}\int_0^{\pi} \sin^3 u\, du = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}.$$

Therefore, by FTC1,

$$\int_0^{\frac{\pi}{2}} \sin^3(2x)\, dx = \Big[2\sin^4 x - \frac{4}{3}\sin^6 x\Big]_0^{\frac{\pi}{2}} = 2 - \frac{4}{3} = \frac{2}{3}. \quad \blacksquare$$

Here is a more challenging example (a bit beyond the scope of this course).

Example. Evaluate $\displaystyle\int \frac{dx}{\sin x + \cos x}$ using substitution.

Solution. Multiply the numerator and denominator by $(\cos x - \sin x)$:

$$\int \frac{dx}{\sin x + \cos x} = \int \frac{\cos x - \sin x}{\cos^2 x - \sin^2 x}\, dx = \int \frac{\cos x - \sin x}{\cos(2x)}\, dx = \underbrace{\int \frac{\cos x}{\cos(2x)}\, dx}_{A} - \underbrace{\int \frac{\sin x}{\cos(2x)}\, dx}_{B}.$$

Now, we evaluate A and B separately, using the double-angle identity $\cos(2x) = 1 - 2\sin^2 x = 2\cos^2 x - 1$.

- For A, we have
$$A = \int \frac{\cos x}{\cos(2x)}\, dx = \int \frac{\cos x}{1 - 2\sin^2 x}\, dx.$$
Let $u = \sin x$. Then $dx = \dfrac{du}{\cos x}$, and so
$$A = \int \frac{\cos x}{1 - 2\sin^2 x}\, dx = \int \frac{\cos x}{1 - 2u^2} \cdot \frac{du}{\cos x} = \int \frac{du}{1 - 2u^2}.$$
Note also that
$$\frac{1}{1 - 2u^2} = \frac{1}{2}\left(\frac{1}{1 + \sqrt{2}u} + \frac{1}{1 - \sqrt{2}u}\right) = \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{1 + \sqrt{2}u} - \frac{-\sqrt{2}}{1 - \sqrt{2}u}\right).$$
Thus,
$$\begin{aligned} A = \int \frac{du}{1 - 2u^2} &= \frac{1}{2\sqrt{2}} \int \left(\frac{\sqrt{2}}{1 + \sqrt{2}u} - \frac{-\sqrt{2}}{1 - \sqrt{2}u}\right) du \\ &= \frac{1}{2\sqrt{2}} \left(\int \frac{\sqrt{2}\, du}{1 + \sqrt{2}u} - \int \frac{-\sqrt{2}\, du}{1 - \sqrt{2}u}\right) \\ &= \frac{1}{2\sqrt{2}} \left(\ln(|1 + \sqrt{2}u|) - \ln(|1 - \sqrt{2}u|)\right) + C \\ &= \frac{1}{2\sqrt{2}} \ln\left(\left|\frac{1 + \sqrt{2}u}{1 - \sqrt{2}u}\right|\right) + C \\ &= \frac{1}{2\sqrt{2}} \ln\left(\left|\frac{1 + \sqrt{2}\sin x}{1 - \sqrt{2}\sin x}\right|\right) + C \end{aligned}$$

- For B, we have
$$B = \int \frac{\sin x}{\cos(2x)}\, dx = \int \frac{-\sin x}{1 - 2\cos^2 x}\, dx.$$
Let $u = \cos x$. Then $dx = -\dfrac{du}{\sin x}$, and so
$$B = \int \frac{-\sin x}{1 - 2\cos^2 x}\, dx = \int \frac{-\sin x}{1 - 2u^2} \cdot \left(-\frac{du}{\sin x}\right) = \int \frac{du}{1 - 2u^2}.$$

But we just saw in the previous case that $\displaystyle\int \frac{du}{1-2u^2} = \frac{1}{2\sqrt{2}} \ln\left(\left|\frac{1+\sqrt{2}u}{1-\sqrt{2}u}\right|\right) + C.$

The difference is that this time, we have $u = \cos x$. So,

$$B = \frac{1}{2\sqrt{2}} \ln\left(\left|\frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right|\right) + C.$$

Now that both A and B are evaluated, we obtain:

$$\int \frac{1}{\sin x + \cos x}\, dx = A - B = \frac{1}{2\sqrt{2}} \ln\left(\left|\frac{(1+\sqrt{2}\sin x)(1-\sqrt{2}\cos x)}{(1-\sqrt{2}\sin x)(1+\sqrt{2}\cos x)}\right|\right) + C. \quad \blacksquare$$

Exercise. If you're curious, simplify the final answer of the previous example to show that:

$$\int \frac{1}{\sin x + \cos x}\, dx = \frac{1}{\sqrt{2}} \ln\left(\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|\right) + C.$$

25.3 Integration by parts

If u and v are both functions of x, then

$$\boxed{\int u\, dv = uv - \int v\, du.}$$

Why? Because of the product formula for the derivative. Recall that:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

So if we integrate both sides with respect to x, we get

$$\int \frac{d(uv)}{\cancel{dx}} \cancel{dx} = \int u \frac{dv}{\cancel{dx}} \cancel{dx} + \int v \frac{du}{\cancel{dx}} \cancel{dx} \Rightarrow \underbrace{\int d(uv)}_{uv} = \int u\, dv + \int v\, du \Rightarrow \int u\, dv = uv - \int v\, du. \checkmark$$

Now, what is Integration By Parts (IBP) good for? It helps with:

- Trading an integral, namely $\int u\, dv$, with another integral, namely $\int v\, du$, in hope for the latter to be simpler!
- Integrating the product of functions: In passing from $\int u\, dv$ to $\int v\, du$, one function, namely u, gets differentiated to du; the other function, namely dv, gets integrated to v.

For example, to evaluate

$$\int x \sin x\, dx;$$

we have two choices: (1) Taking $u = \sin x$ and $dv = x\, dx$, or (2) $u = x$ and $dv = \sin x\, dx$. But in choice (1), u – the function that will be differentiated to $\cos x$ – does not actually become simpler (cos is as complex a thing as sin!). So let's **pick u to be the function that, when differentiated, it actually gets simplified**.

$$u = x \xrightarrow{\text{differentiate}} du = dx \qquad\qquad dv = \sin x\,dx \xrightarrow{\text{integrate}} v = -\cos x$$

$$\int x \sin x\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = x(-\cos x) - \int(-\cos x)\,dx = -x\cos x + \sin x + C. \checkmark$$

In general, there is this **LIATE** rule that often helps: *Take u to be the* <u>*first*</u> *type of function on the list below that appears in the integrand:*

1. ***L**ogarithmic (like* ln*).*
2. ***I**nverse trigonometric.*
3. ***A**lgebraic (like polynomials).*
4. ***T**rigonometric.*
5. ***E**xponential (like e^x).*

And <u>*let dv be everything else*</u>*.*

One last thing: We can also use IBP to evaluate definite integrals:

$$\boxed{\int_a^b u\,dv = \underbrace{[uv]_a^b}_{u(b)v(b)-u(a)v(a)} - \int_a^b v\,du.}$$

Example. Evaluate the following integrals.

(a) $\displaystyle\int \ln x\,dx$.

Solution. Let $u = \ln x$ and $dv = dx$. Then

$$du = \frac{1}{x}dx, \qquad v = x.$$

So,

$$\int \ln x\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = x\ln x - \int 1\,dx = x\ln x - x + C. \quad \blacksquare$$

(b) $\displaystyle\int \frac{\ln x}{x^{10}}\,dx$.

Solution. Let $u = \ln x$ and $dv = x^{-10}\,dx$. Then

$$du = \frac{1}{x}\,dx, \qquad v = \frac{x^{-9}}{-9} = -\frac{1}{9x^9}.$$

So,

$$\int \frac{\ln x}{x^{10}}\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = -\frac{\ln x}{9x^9} - \int\left(-\frac{1}{9x^9}\right)\frac{1}{x}\,dx = -\frac{\ln x}{9x^9} + \frac{1}{9}\int x^{-10}\,dx$$

$$= -\frac{\ln x}{9x^9} - \frac{1}{81x^9} + C. \quad \blacksquare$$

(c) $\displaystyle\int \frac{x^3}{e^x}\,dx$.

Solution. Let $u = x^3$ and $dv = e^{-x}\,dx$. Then

$$du = 3x^2\,dx, \qquad v = -e^{-x}.$$

So,

$$\int x^3 e^{-x}\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = -x^3 e^{-x} + \underbrace{\int 3x^2 e^{-x}\,dx}_{A}.$$

Apply integration by parts again to evaluate A. Let $u = 3x^2$ and $dv = e^{-x}\,dx$. Then

$$du = 6x\,dx, \qquad v = -e^{-x}.$$

So,

$$A = \int 3x^2 e^{-x}\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = -3x^2 e^{-x} + \underbrace{\int 6xe^{-x}\,dx}_{B}.$$

Apply integration by parts again to evaluate B. Let $u = 6x$ and $dv = e^{-x}\,dx$. Then

$$du = 6\,dx, \qquad v = -e^{-x}.$$

So,

$$B = \int 6xe^{-x}\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = -6xe^{-x} + \int 6e^{-x}\,dx = -6xe^{-x} - 6e^{-x} + C.$$

Hence,

$$\int x^3 e^{-x}\,dx = -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + C. \quad \blacksquare$$

(d) $\displaystyle\int \sin(2x) e^{\cos x}\,dx$.

Solution. First we use the substitution $w = \cos x$:

$$\sin(2x) = 2\sin x\cos x, \qquad dx = -\frac{1}{\sin x}\,dw.$$

Then

$$\int \sin(2x) e^{\cos x}\,dx = \int 2\sin x\cos x e^{\cos x}\,dx = \int 2\sin x \cdot w \cdot \left(-\frac{1}{\sin x} dw\right) = -2\int we^w\,dw.$$

Now we evaluate $\displaystyle\int we^w\,dw$ using IBP. Let $u = w$ and $dv = e^w\,dw$. Then

$$du = dw, \qquad v = e^w.$$

So,

$$\int we^w\,dw = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = we^w - \int e^w\,dw = we^w - e^w + C = e^w(w-1) + C.$$

Hence,

$$\int \sin(2x) e^{\cos x}\,dx = -2e^w(w-1) + C = 2e^{\cos x}\big(1 - \cos x\big) + C. \quad \blacksquare$$

(e) $\displaystyle\int \arctan x\,dx$.

Solution. Let $u = \arctan x$ and $dv = dx$. Then

$$du = \frac{1}{1+x^2}\,dx, \qquad v = x.$$

Then

$$\int \arctan x\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

For $\displaystyle\int \frac{x}{1+x^2}\,dx$, we use the substitution $w = 1+x^2$. So $dx = \dfrac{dw}{2x}$, and:

$$\int \frac{x}{1+x^2}\,dx = \int \frac{x}{w}\cdot\frac{dw}{2x} = \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln(1+x^2) + C.$$

Therefore,

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C. \quad \blacksquare$$

(f) $\displaystyle\int e^x\cos x\,dx$.

Solution. Let $I = \displaystyle\int e^x\cos x\,dx$. Here is our first round of IBP:

$$u = \cos x, \qquad dv = e^x\,dx \quad \Rightarrow \quad du = -\sin x\,dx,\ v = e^x.$$

Then

$$I = \int e^x\cos x\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = e^x\cos x + \underbrace{\int e^x\sin x\,dx}_{J}.$$

To evaluate J, we do a second round of IBP:

$$u = \sin x, \qquad dv = e^x\,dx \quad \Rightarrow \quad du = \cos x\,dx, \qquad v = e^x,$$

So,

$$J = \int e^x\sin x\,dx = \int u\,dv \overset{\text{IBP}}{=\!=} uv - \int v\,du = e^x\sin x - \underbrace{\int e^x\cos x\,dx}_{I\text{, again!}}.$$

Hence,

$$I = e^x\cos x + J = e^x\cos x + e^x\sin x - I \quad \Rightarrow \quad I = \frac{e^x}{2}\big(\sin x + \cos x\big) + C. \quad \blacksquare$$

(g) $\displaystyle\int \cos(\ln x)\,dx$.

Solution. Let $t = \ln x$. Then $x = e^t$ and $dx = e^t\,dt$. So,

$$\int \cos(\ln x)\,dx = \int e^t\cos(t)\,dt.$$

We also showed in **(e)** that

$$\int e^t \cos(t)\,dt = \frac{e^t}{2}\big(\sin t + \cos t\big) + C.$$

Returning to $t = \ln x$, we obtain:

$$\int \cos(\ln x)\,dx = \frac{x}{2}\Big(\sin(\ln x) - \cos(\ln x)\Big) + C. \quad \blacksquare$$

(h) $\displaystyle\int_{\frac{1}{e}}^{1} (\ln x)^2\,dx.$

Let $u = (\ln x)^2$ and $dv = dx$. Then

$$du = \frac{2\ln x}{x}dx, \qquad v = x.$$

So,

$$\int_{\frac{1}{e}}^{1} (\ln x)^2\,dx = \int_{\frac{1}{e}}^{1} u\,dv \overset{\text{IBP}}{=\!=\!=} [uv]_{\frac{1}{e}}^{1} - \int_{\frac{1}{e}}^{1} v\,du = \Big[x(\ln x)^2\Big]_{\frac{1}{e}}^{1} - 2\int_{\frac{1}{e}}^{1} \ln x\,dx.$$

We also saw in **(a)** that

$$\int \ln x\,dx = x\ln x - x.$$

Thus,

$$\int_{\frac{1}{e}}^{1} (\ln x)^2\,dx = \Big[x(\ln x)^2\Big]_{\frac{1}{e}}^{1} - 2\Big[x\ln x - x\Big]_{\frac{1}{e}}^{1} = \left(0 - \frac{1}{e}\right) - 2\left(-1 - \frac{2}{e}\right) = 2 + \frac{3}{e}.$$

In particular, we have shown that:

$$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C. \quad \blacksquare$$

26 Average value and area between two curves

26.1 Average value of a function

Recall that the average of finitely many numbers $y_1, \dots, y_n$ is

$$y_{\text{avg}} = \frac{y_1 + \cdots + y_n}{n}.$$

Similarly, for a continuous function f on $[a,b]$, *the average value of f on $[a,b]$* is defined as

$$\boxed{f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.}$$

Example. Find the average value of $f(x) = 1 - x\sqrt{x}$ on $[1,4]$.

Solution. We have

$$f_{\text{avg}} = \frac{1}{4-1}\int_1^4 (1 - x\sqrt{x})\,dx = \frac{1}{3}\left[x - \frac{2}{5}x^{5/2}\right]_1^4 = \frac{1}{3}\big((4 - (64/5)) - (3/5)\big) = -\frac{47}{15}. \quad \blacksquare$$

Example. The temperature throughout a day is modeled by $T(t) = 1 - \cos\!\left(\frac{\pi t}{24}\right)$, in °C, where t is the time in hours since midnight. Find the average temperature from midnight to noon.

Solution. We want the average of T on $[0,12]$:

$$T_{\text{avg}} = \frac{1}{12}\int_0^{12}\left(1 - \cos\!\left(\frac{\pi t}{24}\right)\right)dt = \frac{1}{12}\left[\int_0^{12} 1\,dt - \int_0^{12}\cos\!\left(\frac{\pi t}{24}\right)dt\right].$$

Now,

$$\int_0^{12} 1\,dt = [t]_0^{12} = 12$$

$$\int_0^{12}\cos\!\left(\frac{\pi t}{24}\right)dt = \frac{24}{\pi}\left[\sin\!\left(\frac{\pi t}{24}\right)\right]_0^{12} = \frac{24}{\pi}.$$

Thus

$$T_{\text{avg}} = \frac{1}{12}\left(12 - \frac{24}{\pi}\right) = 1 - \frac{2}{\pi}. \quad \blacksquare$$

26.2 Area between curves

Suppose on $[a,b]$ we have two functions $y_{\text{upper}}(x)$ and $y_{\text{lower}}(x)$ such that $y_{\text{upper}}(x) \geq y_{\text{lower}}(x)$ for all $x \in [a,b]$. Then the area between their graphs from $x = a$ to $x = b$ is

$$\boxed{\int_a^b \big(y_{\text{upper}}(x) - y_{\text{lower}}(x)\big)\,dx.}$$

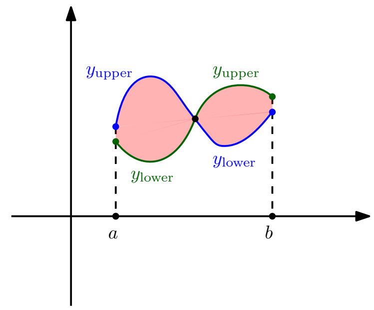
Note that the area under a single curve $y = f(x)$ above the x-axis is the special case $y_{\text{upper}}(x) = f(x)$ and $y_{\text{lower}}(x) = 0$:

$$\int_a^b f(x)\,dx;$$

which is what we already knew!

Given two functions $y_1(x)$ and $y_2(x)$, in order to find the area enclosed between their graphs on an interval $[a, b]$, we need to figure which graph is the upper and which one is the lower. To that end:

1. Find all the solutions to $y_1(x) = y_2(x)$ **inside** $[a, b]$.
2. **For each** interval between two consecutive solutions (or the ends a, b), pick a point c **inside** that interval and compare $y_1(c)$ and $y_2(c)$. The larger one is the upper graph in that interval (and the smaller one is the lower graph in that interval).



Example. Find the area enclosed between $y = 4x$ and $y = x^3$ from $x = 1$ to $x = 3$.

Solution. First, we need to figure the upper and the lower graphs.

$$4x = x^3 \Rightarrow x = -2, 0, 2.$$

The only solution in $[1, 3]$ is $x = 2$. So there are two intervals to check:

$$[1,2] \xrightarrow{c=\frac{3}{2}\in[1,2]} 4c = 6, \quad c^3 = \frac{27}{8} \Rightarrow 4x: \text{ upper}, \quad x^3: \text{ lower}$$

$$[2,3] \xrightarrow{c=\frac{5}{2}\in[2,3]} 4c = 10, \quad c^3 = \frac{125}{8} \Rightarrow 4x: \text{ lower}, \quad x^3: \text{ upper}$$

Thus, the area is

$$\int_1^2 (4x - x^3)\, dx + \int_2^3 (x^3 - 4x)\, dx = \left[2x^2 - \frac{x^4}{4}\right]_1^2 + \left[\frac{x^4}{4} - 2x^2\right]_2^3 = \left(4 - \frac{7}{4}\right) + \left(\frac{9}{4} + 4\right) = \frac{17}{2}. \quad \blacksquare$$

Example. Find the total area enclosed between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \pi$.

Solution. First, we need to figure the upper and the lower graphs.

$$\sin x = \cos x \xrightarrow{x\in[0,\pi]} x = \frac{\pi}{4}.$$

So there are two intervals to check:

$$[0, \tfrac{\pi}{4}] \xrightarrow{c=\frac{\pi}{6}\in[0,\frac{\pi}{4}]} \sin c = \frac{1}{2}, \quad \cos c = \frac{\sqrt{3}}{2} \Rightarrow \cos x: \text{ upper}, \quad \sin x: \text{ lower}$$

$$[\tfrac{\pi}{4}, \pi] \xrightarrow{c=\frac{\pi}{2}\in[\frac{\pi}{4},\pi]} \sin c = 1, \quad \cos c = 0 \Rightarrow \cos x: \text{ lower}, \quad \sin x: \text{ upper}$$

Thus,

$$\text{Area} = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\, dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\, dx = \Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}} + \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\pi}$$

$$= \left(\sqrt{2} - 1\right) + \left(1 + \sqrt{2}\right) = 2\sqrt{2}. \quad \blacksquare$$

We may be asked to find the area enclosed between a *left* curve and a *right* curve, where y is in the interval $[c, d]$. In this case, **we think of x as a function of y**. So we call that left curve $x_{\text{left}}(y)$ and right curve $x_{\text{right}}(y)$. Then, the area is

$$\boxed{\int_c^d \big(x_{\text{right}}(y) - x_{\text{left}}(y)\big)\, dy.}$$

Note that **the integration is with respect to y**.
Similar to the vertical case, given two functions $x_1(y)$ and $x_2(y)$, in order to find the area enclosed between their graphs on an interval $[c, d]$, we need to figure which graph is the right one and which one is the left one. To that end:

1. Find all the solutions to $x_1(y) = x_2(y)$ **inside** $[c, d]$.

2. **For each** interval between two consecutive solutions (or the ends c, d), pick a point a **inside** that interval and compare $x_1(a)$ and $x_2(a)$. The larger one is the right graph in that interval (and the smaller one is the left graph in that interval).

Example. Find the area enclosed between the curves

$$x = y^2 \quad \text{and} \quad x = 2 - y$$

for $y \in [0, 2]$.

Solution. First, we need to figure the right and the left graphs.

$$y^2 = 2 - y \xrightarrow{y \in [0,2]} y = 1.$$

So there are two intervals to check:

$$[0,1] \xrightarrow{a=\frac{1}{2}\in[0,1]} a^2 = \frac{1}{4}, \quad 2 - a = \frac{3}{2} \Rightarrow 2 - y: \text{ right}, \quad y^2: \text{ left}$$

$$[1,2] \xrightarrow{a=\frac{3}{2}\in[1,2]} a^2 = \frac{9}{4}, \quad 2 - a = \frac{1}{2} \Rightarrow 2 - y: \text{ left}, \quad y^2: \text{ right}$$

Thus,

$$\text{Area} = \int_0^1 \big(2-y-y^2\big)\, dy + \int_1^2 \big(y^2+y-2\big)\, dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 + \left[\frac{y^3}{3} + \frac{y^2}{2} - 2y\right]_1^2 = \frac{7}{6} + \frac{11}{6} = 3. \quad \blacksquare$$

27 Volume

We can use integrals to compute *volumes* of the 3-dimensional solids obtained by rotating a region in the plane around an axis.
There are three main methods:

- the disk method;
- the washer method;
- the cylindrical shell method.

The first two methods follow from the same idea: *Slice the solid into infinitely many thin cross-sections, which will be either disks or washers, compute the area of each slice, and add all the areas up using an integral to obtain the volume.*

The third method is a bit different: *Split the solid into infinitely many thin cylindrical shells by "peeling off the shells layer by layer", compute the surface area of each shell, and add all the areas up using an integral to obtain the volume.*

27.1 Disks

Consider the region R below the graph of a function $f(x)$, above the x-axis, and between $x = a$ and $x = b$. Rotate R around the x-axis. This yields a solid and the cross-sections of the solid perpendicular to the x-axis are **disks**.
At a point $x \in [a, b]$, the radius of the disk is $f(x)$, so the area is $A(x) = \pi(f(x))^2$. Adding all these areas up by an integral gives

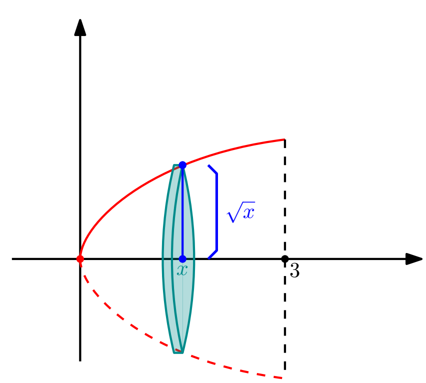
$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi(f(x))^2\,dx.}$$

For rotation around the y-axis (or any other vertical line), we take the analogous steps, only this time we **think of x as a function of y**, and in particular, we **integrate with respect to y**.

Example. Let R be the region between $y = \sqrt{x}$ and the x-axis on $[0, 3]$. Find the volume of the solid obtained by rotating R about the x-axis.

Solution. The radius of the disk on the slice at x is $\sqrt{x}$. Thus,

$$V = \int_0^3 \pi(\sqrt{x})^2\,dx = \pi\int_0^3 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^3 = \frac{9\pi}{2}. \quad \blacksquare$$

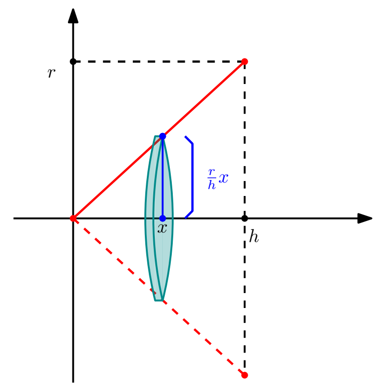


Example. Prove that the volume of a cone of height h and radius r is $\dfrac{1}{3}\pi r^2 h$.

Solution. Let R be the region between $y = \frac{r}{h}x$ and the x-axis on $[0, h]$. Then the solid obtained by rotating R about the x-axis is a cone of height h and radius r. The radius of the disk on the slice at x is $\frac{r}{h}x$. Thus,

$$V = \int_0^h \pi \left(\frac{r}{h}x\right)^2 dx = \frac{\pi r^2}{h^2} \int_0^h x^2 \, dx$$

$$= \frac{\pi r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \frac{1}{3}\pi r^2 h. \quad \blacksquare$$

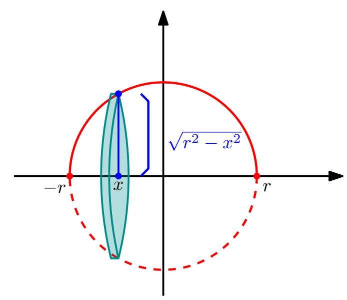


Example. Prove that the volume of a sphere of radius r is $\dfrac{4}{3}\pi r^3$.

Solution. Let R be the region between $y = \sqrt{r^2 - x^2}$ and the x-axis on $[-r, r]$. Then the solid obtained by rotating R about the x-axis is a sphere of radius r. The radius of the disk on the slice at x is $\sqrt{r^2 - x^2}$. Thus,

$$V = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} (r^2 - x^2)\, dx = \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi \left(\left(r^3 - \frac{r^3}{3}\right) - \left(-r^3 + \frac{r^3}{3}\right)\right) = \frac{4}{3}\pi r^3. \quad \blacksquare$$



27.2 Washers

Consider the region R between the graphs of two functions $y_{\text{upper}}(x)$ and $y_{\text{lower}}(x)$ between $x = a$ and $x = b$. Rotate R around the x-axis. This yields a solid and the cross-sections of the solid perpendicular to the x-axis are **washers**.
At a point $x \in [a, b]$, the outer radius of the washer is $y_{\text{upper}}(x)$ and the inner radius of the washer is $y_{\text{lower}}(x)$. So the area is $A(x) = \pi(y_{\text{upper}}(x))^2 - \pi(y_{\text{lower}}(x))^2$. Adding all these areas up by an integral gives

$$\boxed{V = \int_a^b A(x)\, dx = \int_a^b \pi \left((y_{\text{upper}}(x))^2 - (y_{\text{lower}}(x))^2\right) dx.}$$

Note that we may need to figure out which graph is actually the upper or the lower, as we did for computing the area between the two graphs.

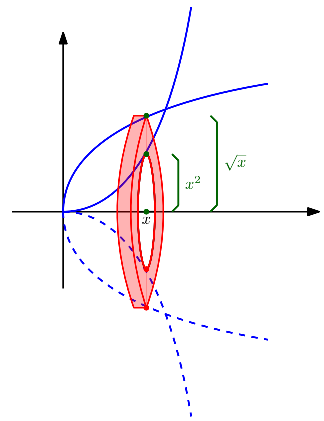
For rotation around the y-axis (or any other vertical line), we take the analogous steps, only this time we **think of x as a function of y**, and in particular, we **integrate with respect to y**. Again, we may need to figure out which graph is actually the right or the left one, as we did for computing the area between the two graphs.

Example. Let R be the region between the graphs of $y = x^2$ and $y = \sqrt{x}$. Find the volume of the solid obtained by rotating R around the x-axis.

Solution. Note that R is enclosed by $x = 0$ and $x = 1$, and in the interval $[0, 1]$, it is easily seen that $\sqrt{x}$ has the upper graph and x^2 has the lower graph. Thus,

$$V = \int_0^1 \pi \left((\sqrt{x})^2 - (x^2)^2 \right) dx = \pi \int_0^1 (x - x^4)\, dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{3\pi}{10}. \quad \blacksquare$$

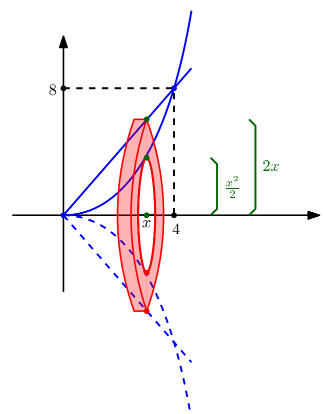


Example. Let R be the region between the graphs $y = 2x$ and $y = \dfrac{x^2}{2}$. Set up the integral that gives the volume of the solid obtained by rotating R around:

(a) The x-axis.

Solution. Note that R is enclosed by $x = 0$ and $x = 4$, and in the interval $[0, 4]$, it is easily seen that $y = 2x$ has the upper graph and $\frac{x^2}{2}$ has the lower graph. Thus,

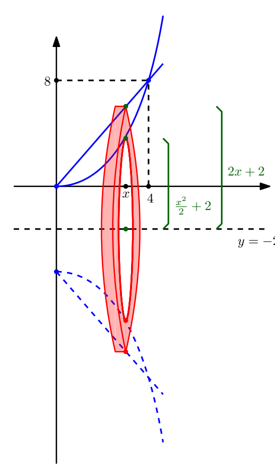
$$V = \int_0^4 \pi \left((2x)^2 - \left(\frac{x^2}{2} \right)^2 \right) dx. \quad \blacksquare$$



(b) $y = -2$.

Solution. Since $y = -2$ is **below** both graphs, the outer radius of the washer is $2x - (-2) = 2x + 2$ and the inner radius of the washer is $\frac{x^2}{2} - (-2) = \frac{x^2}{2} + 2$. Thus,

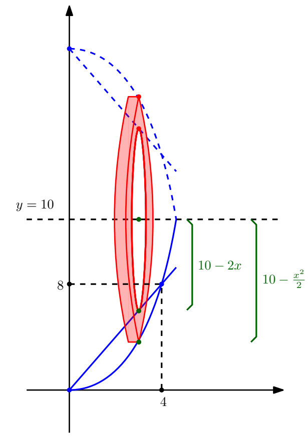
$$V = \int_0^4 \pi \left((2x+2)^2 - \left(\frac{x^2}{2} + 2 \right)^2 \right) dx. \quad \blacksquare$$



(c) $y = 10$.

Solution. Since $y = 10$ is **above** both graphs, the outer radius of the washer is $10 - \frac{x^2}{2}$ and the inner radius of the washer is $10 - 2x$. Thus,

$$V = \int_0^4 \pi \left(\left(10 - \frac{x^2}{2} \right)^2 - (10 - 2x)^2 \right) dx. \quad \blacksquare$$



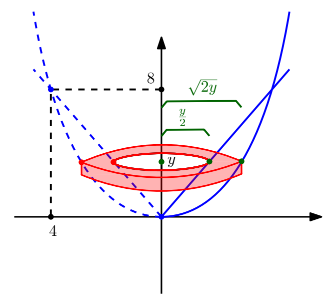
(d) The y-axis.

Solution. Since the rotation is around a **vertical** line, we view x as a function of y:

- The right function is $x = \sqrt{2y}$ and the left one is $x = \frac{y}{2}$;
- R is enclosed between the two graphs for $y \in [0, 8]$.

Since the y-axis is to the left of both graphs, the outer radius of the washer is $\sqrt{2y}$, and the inner radius is $\frac{y}{2}$. Thus,

$$V = \int_0^8 \pi \left(\left(\sqrt{2y} \right)^2 - \left(\frac{y}{2} \right)^2 \right) dy. \quad \blacksquare$$



A **summary** of the disk and the washer method:

- Disk: $V = \displaystyle\int \pi (\text{radius of the disk})^2$.
- Washer: $V = \displaystyle\int \pi \left((\text{outer radius of the washer})^2 - (\text{inner radius of the washer})^2 \right)$.
- Rotation around a **horizontal** line: Express as functions of $\boldsymbol{x}$ and integrate w.r.t. $\boldsymbol{x}$.
- Rotation around a **vertical** line: Express as functions of $\boldsymbol{y}$ and integrate w.r.t. $\boldsymbol{y}$.

27.3 Cylindrical shells

Consider the region R in the plane and rotate R around a horizontal or vertical line. This yields a solid, and in order to compute its volume, this time we split it into cylindrical shells that revolve around the axis of rotation. The surface area of such a shell of height h and radius r is $A = 2\pi r h$. Adding all these areas up by an integral gives

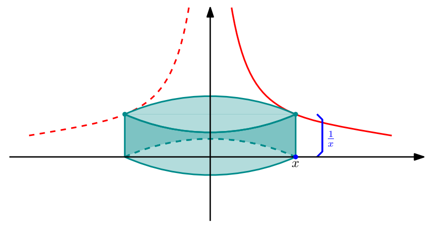
$$\boxed{V = \int A = \int 2\pi r h.}$$

This method can be used for horizontal and vertical rotations and for rotating the region between the graphs of two functions. For each problem, we need to determine how r and h are functions of x or y, and then integrate with respect to x or y correspondingly.

Example. Let R be the region between the graph of $y = \frac{1}{x}$ and the x-axis for $x \in [1, 5]$. Find the volume of the solid obtained by rotating R around the y-axis.

Solution. Here, the radius of the shell is $r = x$, the height of the shell is $h = \frac{1}{x}$, and x runs from 1 to 5. Thus,

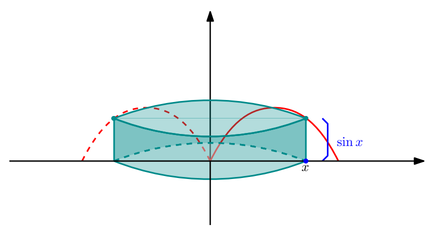
$$V = \int_1^5 2\pi x \left(\frac{1}{x}\right) dx = 2\pi \int_1^5 dx = 2\pi \, [x]_1^5 = 8\pi. \quad \blacksquare$$



Example. Let R be the region between the graph of $y = \sin x$ and the x-axis for $x \in [0, \pi]$. Using cylindrical shells, set up an integral that evaluates the volume of the solid obtained by rotating R around the y-axis.

Solution. Here, the radius of the shell is $r = x$, the height of the shell is $h = \sin x$, and x runs from 0 to π. Thus,

$$V = \int_0^\pi 2\pi x \sin x \, dx. \quad \blacksquare$$



Example. Let R be the region between the graphs of $y = \frac{x}{3}$ and $y = \sqrt{x}$. Using cylindrical shells, set up an integral that evaluates the volume of the solid obtained by rotating R around:

(a) The y-axis.

Solution. Note that R is enclosed by $x = 0$ and $x = 9$ (roots of $\frac{x}{3} = \sqrt{x}$), and in the interval $[0, 9]$, it is easily seen that $y = \sqrt{x}$ has the upper graph and $y = \frac{x}{3}$ has the lower graph. It follows that here, the radius of the shell is $r = x$, the height of the shell is $h = \sqrt{x} - \frac{x}{3}$, and x runs from 0 to 9. Thus,

$$V = \int_0^9 2\pi x \left(\sqrt{x} - \frac{x}{3}\right) dx. \quad \blacksquare$$

(b) $x = -2$.

Solution. Since $x = -2$ is to the **left** of both graphs, here, the radius of the shell is $r = x - (-2) = x + 2$, the height of the shell is $h = \sqrt{x} - \frac{x}{3}$, and x runs from 0 to 9. Thus,

$$V = \int_0^9 2\pi(x + 2) \left(\sqrt{x} - \frac{x}{3}\right) dx. \quad \blacksquare$$

(c) $x = 10$.

Solution. Since $x = 10$ is to the **right** of both graphs, here, the radius of the shell is $r = 10 - x$, the height of the shell is $h = \sqrt{x} - \frac{x}{3}$, and x runs from 0 to 9. Thus,

$$V = \int_0^9 2\pi(10 - x) \left(\sqrt{x} - \frac{x}{3}\right) dx. \quad \blacksquare$$

(d) The x-axis.

Solution. Since the rotation is around a **horizontal** line, we view x as a function of y:

- The upper graph belongs to $x = y^2$ and the lower graph belongs to $x = 3y$;
- R is enclosed between the two graphs for $y \in [0, 3]$.

Since the x-axis is below both graphs, here, the radius of the shell is $r = y$, the height of the shell is $h = 3y - y^2$, and y runs from 0 to 3. Thus,

$$V = \int_0^3 2\pi y \left(3y - y^2\right) dy. \quad \blacksquare$$

A **summary** of the cylindrical shell method:

- $V = \displaystyle\int A = \int 2\pi r h.$
- Rotation around a **horizontal** line: Express as functions of $\boldsymbol{y}$ and integrate w.r.t. $\boldsymbol{y}$.
- Rotation around a **vertical** line: Express as functions of $\boldsymbol{x}$ and integrate w.r.t. $\boldsymbol{x}$.

28 Work

Suppose a constant force F is applied to an object and moves it from $x = a$ to $x = b$. In this case the work done on the object is

$$\text{Work} = \text{Force} \times \text{Distance} = F \times (b - a).$$

But what if the force is *not* constant, and instead depends on the position x? For instance, imagine lifting a leaking bucket: as water leaks out, the weight decreases, so does the force required.

If the force, as a function of position, is $F(x)$, then the work done from position $x = a$ to position $x = b$ is:

$$\text{Work} = \int_a^b F(x)\,dx.$$

Example. An 8 m chain of mass 32 kg with constant density hangs from the top of a building. How much work is required to lift the entire chain to the top? (Acceleration due to gravity is $10\,\mathrm{m/s^2}$.)

Solution. The chain has constant density $\frac{32}{8} = 4$ kg/m. So after lifting x meters, the remaining $(8 - x)$ meters hanging have mass

$$M(x) = 4(8 - x) \text{ kg}.$$

Hence, the force needed at that moment is

$$F(x) = M(x) \cdot 10 = 40(8 - x).$$

Thus, the total work is

$$W = \int_0^8 40(8 - x)\,dx = 40\left[8x - \frac{x^2}{2}\right]_0^8 = 40(64 - 32) = 1280 \text{ J.} \quad \blacksquare$$

Example. Rapunzel's hair has constant density 4 kg/m and hangs 200 m down her tower. Set up the integrals that evaluate the work in each of the following cases. (Again, acceleration due to gravity is $10\,\mathrm{m/s^2}$.)

(a) Lifting the hair all the way up.

Solution. After lifting x meters, the remaining $200 - x$ meters of hair have mass

$$M(x) = 4(200 - x) \text{ kg}.$$

Thus, the force required is

$$F(x) = 40(200 - x);$$

and the work is

$$W = \int_0^{200} 40(200 - x)\,dx. \quad \blacksquare$$

(b) Lifting the hair only halfway up.

Solution. Simply change the upper limit to 100:

$$W = \int_0^{100} 40(200 - x)\,dx. \quad \blacksquare$$

(c) Same as **(a)**, but a 5 kg Uber Eats order is tied to the end.

Solution. After lifting x meters, the remaining $200 - x$ meters of hair, along with the food order, have mass

$$M(x) = 4(200 - x) + 5 = 805 - 4x \text{ kg}.$$

Thus, the force required is

$$F(x) = 10(805 - 4x);$$

and the work is

$$W = \int_0^{200} 10(805 - 4x)\,dx. \quad \blacksquare$$

(d) Same as (c), but the food order falls off after being lifted 115 m.

Solution. This is in two parts. For the first 115 meters, the setup is as in **(c)**, and so the work is

$$W_1 = \int_0^{115} 10(805 - 4x)\,dx.$$

For the remaining 85 meters, with the food fallen off, the setup is as in **(a)**, and so the work is:

$$W_2 = \int_{115}^{200} 40(200 - x)\,dx.$$

Thus, the total work is:

$$W = \int_0^{115} 10(805 - 4x)\,dx + \int_{115}^{200} 40(200 - x)\,dx. \quad \blacksquare$$

Example. A 50 m rope of density 2 kg/m hangs into a well. A bucket of mass 10 kg is attached, initially containing 6 kg of water. But there is a leak of constant rate in the bucket, and when it reaches the top, only 1 kg of water remains. Compute the work done lifting the bucket. (Assume the acceleration due to gravity is 10 m/s^2.)

Solution. Water lost: $6 - 1 = 5$ kg over 50 m. Therefore, the leakage rate is $\frac{5}{50} = \frac{1}{10}$ kg/m. It follows that the water mass after lifting x meters is:

$$M_{\text{water}}(x) = 6 - \frac{x}{10} \text{ kg}.$$

The remaining $50 - x$ meters of rope, of constant density 2 kg/m, has mass:

$$M_{\text{rope}}(x) = 2(50 - x) = 100 - 2x \text{ kg}.$$

Recall that the bucket itself has mass 10 kg. This means the total mass after lifting x meters is:

$$M(x) = M_{\text{water}}(x) + M_{\text{rope}}(x) + 10 = 6 - \frac{x}{10} + 100 - 2x + 10 = 116 - \frac{21}{10}x \text{ kg}.$$

Thus, the force required is

$$F(x) = 10\left(116 - \frac{21}{10}x\right) = 1160 - 21x;$$

and the total work is:

$$W = \int_0^{50} (1160 - 21x)\,dx = \left[1160x - \frac{21x^2}{2}\right]_0^{50} = 58000 - 26250 = 31750 \text{ J}. \quad \blacksquare$$